STUDY REPORT
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PROBABILITY MODELS FOR THEATER NUCLEAR WARFARE

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This paper proposes specific probabilistic approaches to address several major problems associated with the representation of tactical nuclear warfare at the theater level.

The first problem is identifying the locations of small units (potential nuclear targets) such as companies or battalions within theater-level conventional scenarios or model outputs. Current approaches to identifying these small unit locations fail to take into account the variability that might be realized in any specific battle. A two-dimensional multivariate model is proposed to describe uncertainty about the precise location of the potential targets. As targets may be aggregated and/or precluded from fires due to collateral damage and other constraints, the multivariate location model is suitably modified to indicate the two-dimensional distributions of possible weapon aimpoint locations. The research also incorporates probability models of target acquisition and target location error.

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The probability models described in this paper may be used as a research tool to estimate the sensitivity of exchange outcomes to various data and assumptions, as a surrogate for detailed, complex simulation models; or as an estimator of the sample space of all possible outcomes of a theater nuclear exchange.

PROBABILITY MODELS FOR

THEATER NUCLEAR WARFARE

September 1989

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THE RESEARCH SPONSOR was the Director, US Army Concepts Analysis Agency (CAA).

THE OBJECTIVE OF THE RESEARCH was to develop a probability model for theater-level tactical nuclear warfare based on a probabilistic force array.

THE MAIN ASSUMPTIONS used in this research were:

(1) The uncertainty about the actual location of units on some future battlefield can be described using multinormal probability distributions, dependent between all X coordinates and between all Y coordinates for all units, with the X coordinates assumed to be mutually independent of all Y coordinates.

(2) Aggregation and bonus effects can be evaluated for pairs of units only.

(3) Various mixing parameters can be approximated as constants.

(4) DGZ shifts due to preclusion can be modeled from the single preclusion area most likely to cause a shift.

(5) Distributions can be evaluated using standard approximations.

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(6) Assumptions standard in nuclear effects models relating to the nuclear weapon and the target can be used. These assumptions are detailed within the text.

(7) The unit defeat probabilities can be evaluated independently.

THE BASIC APPROACH used in this research was to use multivariate probability distributions to describe uncertainty about the precise location of the potential targets. As targets may be aggregated and/or precluded from fires due to collateral damage and other constraints, the multivariate location model is suitably modified to indicate the two-dimensional distributions of possible weapon aimpoint locations. The research also incorporates probability models of target acquisition and target location error. From these distributions, it is possible to determine the probability that a targetable subunit (such as a company or battery) can be defeated and the space of all possible outcomes of a tactical nuclear exchange (in terms of the defeat or failure to defeat a unit) can be specified.

THE PRINCIPAL FINDING of the research is that is is possible to develop an analytic probability model of a theater-level tactical nuclear exchange.

THE RESEARCH WAS PERFORMED BY MAJ Mark A. Youngren.

COMMENTS AND QUESTIONS may be sent to the Director, US Army Concepts Analysis Agency, ATTN: CSCA-RQR, 8120 Woodmont Avenue, Bethesda, MD 20814-2797.

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CHAPTER 1

INTRODUCTION

Section I. BACKGROUND

Modeling Nuclear Warfare at the Theater Level

The US Army Concepts Analysis Agency (CAA) has the responsibility of conducting analysis of issues of concern to the Department of the Army at the theater level. One such issue is the possible employment of tactical nuclear weapons - that is, nuclear weapons employed by combined forces against military targets within a theater.

The number of nuclear weapons available for employment within a theater is relatively small, yet the potential impact of each round is great. As a result, tactical nuclear weapon employment is modeled at a round-by-round and target-by-target degree of resolution. Furthermore, the use of tactical nuclear weapons may be greatly influenced by the locations of potential targets. The use of tactical nuclear weapons is often precluded to avoid damage or injury to nearby civilian populations and friendly troops. Targets may also be aggregated in order to damage two or more units with a single weapon. Potential nuclear targets include units a small as companies, batteries and individual missile launchers, thus a tactical nuclear model will need to have input relating to the characteristics and precise locations of these potential target units.

Models of tactical nuclear warfare within a theater rely on output from a theater-level conventional model or scenario to define the battlefield situation at the time at which nuclear weapons may be employed. Theater-level models and scenarios generally model units only at the division level or higher, often in fairly large discrete time steps (12 to 24 hours). As a result, the locations and actions of potential target units will not be represented in the outputs from these sources. Even if model or scenario results were available that tracked units at the required degree of resolution in time and space, the locations and actions of small units could easily vary within the same overall theater scenario. Therefore, we must regard the detailed actions and locations of potential target units as not merely unknown (given typical theater scenarios), but uncertain; an analysis highly dependent upon unit locations (such as that based on tactical nuclear models) will need to consider the effects of such uncertainty.

This paper proposes a model for representing a laydown of nuclear weapons at the theater level. It begins by developing probability models to reflect the uncertainty inherent in representations of the activities and locations of potential nuclear targets in the context of any general theater scenario. These models can provide an analytic or Monte Carlo solution to the effect of a nuclear exchange upon all potential targets on both sides. This solution may be used directly to support analyses related to the use of nuclear weapons in a theater, or it may be used to determine the possible outcomes of an exchange to construct inputs to a theater-level conventional simulation.

Several models of conventional warfare exist at the theater level. Two models used at CAA are called the Concept Evaluation Model (CEM) and the Force Evaluation Model (FORCEM). FORCEM has several important characteristics it shares with other theater-level simulations. First, it is a low resolution model, representing combat forces at the division and higher level. This level of resolution is common for theater-level models and scenarios. Second, it is a deterministic, *expected value* model. As an expected value model, it requires a single input data set, which represents the mean or expected value of various stochastic processes, and produces a single output set. Unfortunately, an expected value model does not produce an expected value output: the use of this type of model creates special concerns in properly handling the uncertainty inherent in the input data. In this paper, we assume that we have FORCEM outputs available that define the initial theater-level situation at the time that nuclear weapons may be employed. However, the techniques presented herein may be used with any general theater-level model or scenario.

Nuclear Force Arrays

A *nuclear force array* is a set of coordinates specifying the actual locations of combat units that may be potential nuclear targets within some theater-level model or scenario. A *unit* is a military organization composed of personnel and equipment that may be killed or destroyed by nuclear weapons. Nuclear *targets* are units that are planned for engagement by tactical nuclear weapons. *Relocatable targets* are units which have the capability to move during the scenario of interest (although they may or may not retain mission capability during movement). Almost all of the combat units are potential relocatable targets. As a consequence of this movement capability, units do not remain in a static array; the force array will change over time in accordance with the overall theater-level model or scenario.

When a combat simulation is run at a resolution sufficient to represent each potential nuclear target unit, it is important to correctly array the various units at the start of the simulation. We normally assume that tactical nuclear weapons will be employed only after some period of theater-level conventional combat. Thus the starting point for a nuclear simulation will be at some point in time after the initiation of conflict. Starting positions for the nuclear simulation, representing unit locations at some point during a conventional conflict, may be generated from a large-scale conventional simulation, such as a theater combat model (e.g., FORCEM), or an established scenario (which itself is frequently generated from a wargame or larger simulation). In either case, the model output or scenario will frequently only specify the center of mass location of the major combat formations (such as divisions), without disaggregation into smaller units (such as maneuver and logistics companies).

Even in situations where the unit locations are specified to the degree of resolution desired, it is important to realize that the locations of combat units in some future conflict cannot be known with certainty (if we omit the trivial case of current peacetime deployment). As a result, any specific model output or scenario specification must be regarded as a single realization of the set of random variables describing all possible locations of the combat units.

Tactical nuclear weapons are employed against military targets in a theater of war to achieve a tactical or operational objective. Tactical nuclear weapons, particularly small yield weapons fired by artillery, may be employed against mobile units as small as a maneuver company, artillery firing section, or missile launcher. On a theater scale, the number of such potential targets is on the order of 10^4. Most theater-level models, such as FORCEM, and theater-level scenarios provide locations only for division-sized or specialized units. The generation of company-level locations across the theater, given typical theater scenario resolution, is a formidable task.

Current Arraying Practice

Currently, two methods are employed to specify the location of smaller level units prior to running a high resolution nuclear simulation. The first method is to have a subject matter expert (such as a military officer with doctrinal knowledge and field experience) manually array the units on a map. Although this method generates an array that is realistic, it may take an officer several months, working full time, to generate an array for a two-sided theater exchange. Also, this array is only a single realization of a set of random variables, as any other equally qualified officer (or even

the same officer at a different time) will generate an array that is most likely similar but nevertheless different in many specific details.

Another approach that is currently used is to generate a *template* consisting of stylized unit locations based on doctrine. This template will specify a doctrinally correct set of relationships between units, but is sure to differ from any specific array that may be generated for actual forces deployed on actual terrain.

Both approaches fail to take into account the variations that will occur between any specific force arrays that may be generated (or which may occur in some future conflict). The first method permits alternative arrays to be generated, but this can be done only with great time and expense, and requires separate runs of a simulation to account for the impact of the array variations. The second (templating) method cannot be used to generate any different arrays (unless doctrine or the forces represented are changed).

Our Approach

It is our thesis that uncertainty may only properly be described using a probability distribution. Since the unit locations in some future conflict are unknown, we must specify a probability distribution to capture that uncertainty. Before manual arraying, a subject matter expert will explicitly or implicitly examine the force structure, mission, and doctrine of each side to establish a doctrinal relationship among the units on each side. This set of doctrinal relationships is basically the same as the template that might be developed for that force. The subject matter expert will then shift the units from these doctrinal locations based on the terrain, unit missions, etc. Shifts in one unit will likely cause shifts in adjacent unit locations to avoid "gaps" and "overlaps."

Our probability model is based on this representation of an expert arrayer. We begin with a doctrinal template that establishes the most likely location for the unit. This template specifies the mean vectors for the coordinates of the units in the array. Shifts from this most likely value based on the terrain, unit missions, etc. are accounted for through variances for each unit location. Shifts in adjacent unit values are accounted for using covariances for each pair of units. A multinormal distribution for the distances across the width of the template (generally parallel to the FLOT) and another multinormal distribution for the depth of the template (generally perpendicular to the FLOT) with the appropriate means, variances, and covariances is used to describe our uncertainty about the unit locations. Details pertaining to this model are given in Section V.

An example of a template for a notional Red division is shown in Figure 1, below. The template overlaid with isoprobability contours is shown in Figure 2.
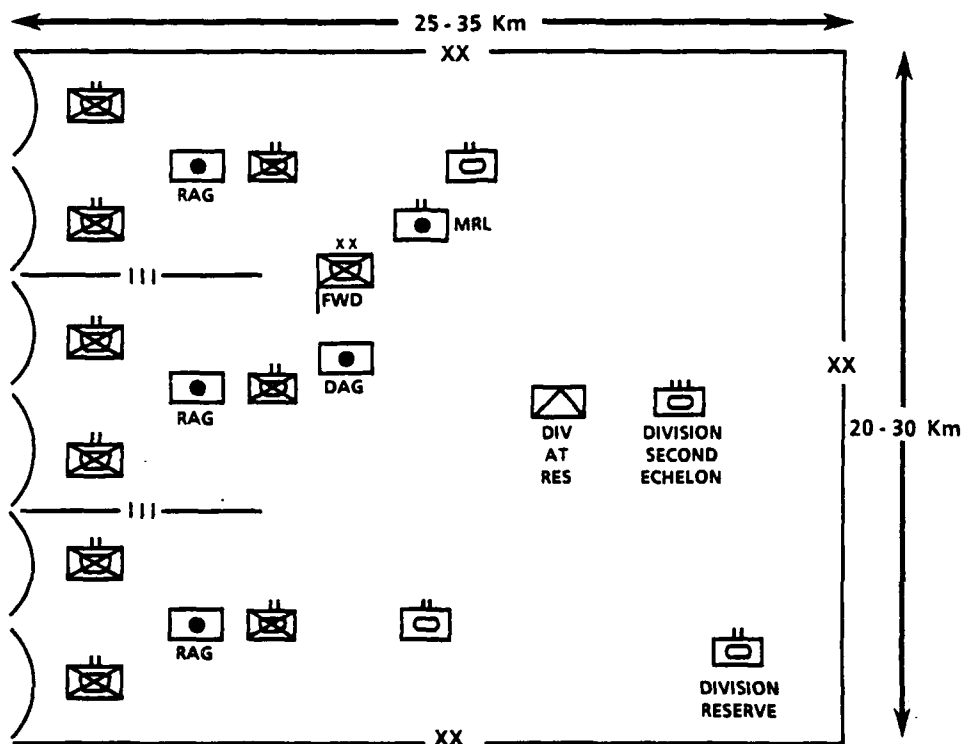
Figure 1. Example of a Template for a Red Division
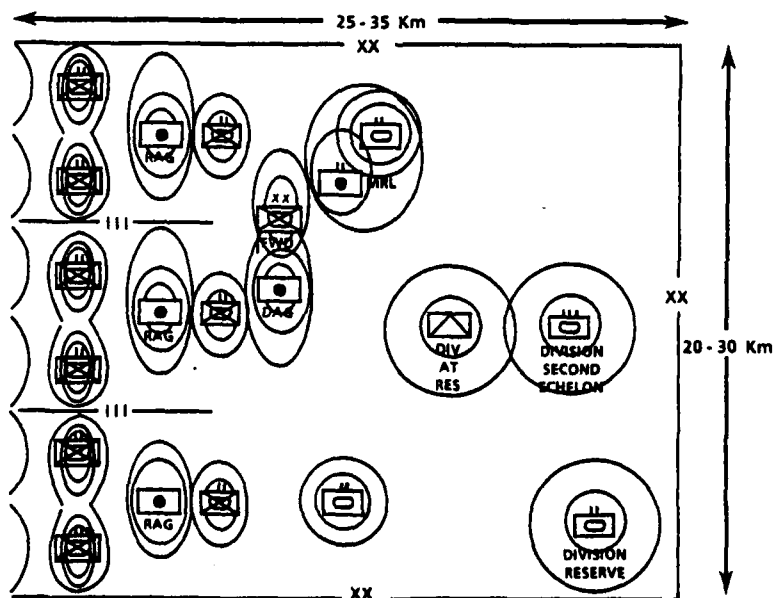
Figure 2. Example of a Template for a Red Division with Isoprobability Contours

Templates are established for each unit that is represented in the theater model or scenario (usually divisions). Since the templates are based on doctrine, many units will share the same template. The coordinates for each unit subordinate to the templated unit are specified relative to the center of mass (or for convenience the front right corner) of the larger unit; these relative coordinates can easily be converted to terrain based coordinates if desired. The X-axis is therefore across the front of the unit (generally parallel to the FLOT if the unit is in the front lines) and the Y-axis is perpendicular to the front of the unit (generally measured as a distance from the FLOT if the unit is in the front lines) (Figure 3).
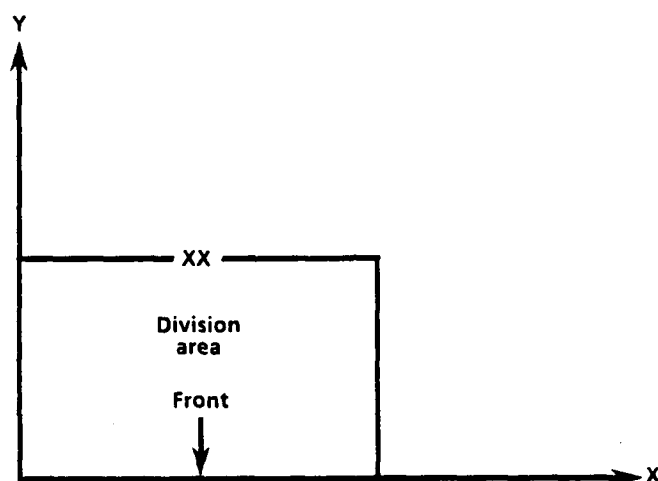
Figure 3. Orientation of the X and Y axis for a Division Template

The parameters of the prior distribution for all units will be based on expert opinion. Posterior distributions for the parameters will be based on data collected from a system known as the Nuclear Fire Planning and Assessment Model Graphical Analysis Package (NUFAM-GAP). The procedure for doing this is discussed in more detail in Section V.

Determining the Impact of the Probabilistic Force Arrays on the Nuclear Exchange

The actual and perceived locations of acquired units, relative to each other and relative to areas from which nuclear weapons are precluded, will dictate what nuclear weapons may be employed to engage the potential target units. Nuclear weapons will be employed as dictated by the operational and strategic situation, with the intent of causing a sudden and dramatic change in the conduct of the battle. The commander of the forces on a side will have an overall objective (such as stabilizing the Forward Line of Own Troops - FLOT - in the defense or achieving a breakthrough in the

offense) that will necessitate the use of nuclear weapons. In order to meet this objective, the commander will specify the *defeat criteria* against each unit - that is, the necessary degree of damage to be achieved against each unit to meet his objective. The defeat criteria will differ from unit to unit depending upon the unit mission, the posture, the equipment, etc. The criteria applied to larger units (such as divisions) will frequently focus fires on critical units. For example, the defeat criteria for a unit might be achieving a latent lethal dose (about 450 rad) against at least 50% of the personnel in the unit. The defeat criteria for a particular division might be to defeat at least 50% of the infantry units *or* at least 40% of the armor units in the division.

Nuclear weapons will be employed to maximize the probability of defeating as many units as possible. Planners will specify a *desired ground zero (DGZ)* for each weapon (an aimpoint) that will meet the defeat criteria for one or more units subject to any constraints placed upon the employment of the weapons. National policy for nuclear weapons employment will normally preclude the use of the weapons in areas that will cause civilian casualties, casualties to friendly troops, etc. We refer to these policy constraints as *preclusion* and an area within which a nuclear weapon cannot be employed as a *preclusion area.* Ideally, the DGZ for a weapon employed against a single target unit would be the center of mass of that unit. The existence of preclusion areas will often force the planner to shift the DGZ away from the unit center of mass. The DGZ may also be shifted to cause the weapon to cover two or more units with effects sufficient to meet the defeat criteria simultaneously. We refer to combining target units into a single target for a weapon as *target aggregation* or simply *aggregation.*

The closer units are together, the more likely they can be engaged as an aggregate target. The closer they are to preclusion areas, the more likely they will be engaged with a shifted DGZ or not engaged at all. Since the locations (both actual and perceived) of the units are random, limitations on the type and locations of the weapons that may be employed resulting from preclusion and aggregation, and the effects that can be realized against the targets, are also random. We can combine our distributions for each unit location, errors in target location and weapon delivery, and the possibilities of aggregation and preclusion, into a probability for each unit that it may be defeated with a particular nuclear weapon. The impact of the defeat or failure to defeat each unit can be combined across the theater to determine the possible outcomes of a theater-level exchange of tactical nuclear weapons.

Section II. THE MODEL

Overview

The model developed within this paper uses the probabilistic force arrays as a fundamental building block to determining the possible outcomes of a tactical nuclear exchange. We start with the idea of using a distribution to describe the actual ground location of a targetable unit at the time that nuclear weapons may be employed. Targeting will actually be performed upon units that are found by some sensor, identified and retained as valid targets (we use the term *available* to describe units meeting these criteria) through some target acquisition process. Associated with the acquisition means is some random target location error, which we describe with a distribution dependent on the numbers and types of sensors available for target acquisition. The location actually used for targeting is the *perceived* location; the distribution of the perceived location is found by combining the distribution of the actual location with the appropriate target location error distribution.

If a targetable unit is available, we consider it as a possible target for each weapon that is potentially available that can be used against that type of target (at a later step in the process, we will allocate available weapons against potential targets). If there were no constraints upon the use of tactical nuclear weapons and we used a weapon for each target, each weapon would be aimed at the center of the associated perceived target location. However, targets may be aggregated; that is, a single weapon may be used against two or more targets. In this model, we consider only aggregate target pairs; it is relatively unusual in practice for 3 or more targets to be available for aggregation. If a weapon is used against two targets, the aimpoint (DGZ) will be directed against a point between the two targets.

Nuclear weapons use may also be constrained by requiring that they avoid (preclude) creating effects within preclusion areas. If these constraints are applicable, an available target may be precluded from fire with a particular weapon if there is no way it can be engaged without causing unacceptable damage in one or more preclusion areas. In other cases, it may be possible to engage the target without causing unacceptable damage in any preclusion areas by shifting the DGZ away from target center (or directly between two targets if aggregated) in a direction away from the closest preclusion area.

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Thus we have 5 possible outcomes for each unit / weapon pair. An available unit may be:

(1) Engaged as a single target without a DGZ shift caused by preclusion,

(2) Engaged as a single target with a DGZ shift caused by preclusion,

(3) Engaged as part of an aggregate target pair without a DGZ shift caused by preclusion,

(4) Engaged as part of an aggregate target pair with a DGZ shift caused by preclusion, or

(5) Precluded from engagement.

These outcomes all have an associated probabilities of occurrence based on the distribution of the perceived target locations and their separation from various preclusion areas.

Given a set of available weapons, range and target preference considerations, it is possible to determine the set of weapons that may be fired at some subset of the available nonprecluded targets. If we look at all of the possible available nonprecluded target sets, we can determine the probability that a given weapon type may be used against each unit.

If a particular weapon type is fired at a unit, the weapon will be aimed at the DGZ (which may be shifted as explained previously). Each weapon type has an associated random accuracy; thus the distribution of the point of detonation (*Actual Ground Zero* or AGZ) will be a combination of the distributions for the perceived target location and the weapon accuracy. We end up with a distribution for the AGZ associated with each of the four engagement possibilities where weapons are fired; from this we can compute the distribution of the distance between the AGZ and the actual unit location (from which we can determine the probability that the unit is defeated using the weapon) for each unit, each weapon, and each engagement possibility. We can combine across the weapons and engagement possibilities to get the probability of defeat for each unit represented in our model.

The paper simply derives the distributions and probabilities associated with each step of the process described above. We start with a set of targetable units whose locations are described using multivariate probability distributions, and end up with a probability of defeating each unit in the set. From this, we can determine all possible outcomes of any nuclear exchange with the associated probabilities. The paper is divided into four major parts. The first part (Chapter 1) provides background and some of the details behind the scenario. Also discussed in Chapter 1 is the model used to determine unit availability probabilities. The second part (Chapter 2) focuses on determining the possible engagement outcomes and the associated DGZ and AGZ distributions. The third part (Chapter 3) discusses the how to combine the distributions top determine the possible outcomes of a

nuclear exchange. The final part, consisting of the of a summary chapter (Chapter 4) and Appendices, provides additional technical detail.

Some of the subjects discussed are the topic of several separate research papers. When this occurs, an overview of the topic is provided and the specific papers are referenced for more detail (Youngren [1989a,b,c]).

Uses of the Model

The probability model for nuclear force arrays described in this paper may be used in three distinct ways: as a means for selecting specific nuclear exchange outcomes that may be simulated in low-resolution, expected value, conventional theater-level simulations such as FORCEM; as a research tool to determine the sensitivity of various outcomes to different assumptions and sets of input parameters; and as a surrogate for detailed simulation models of nuclear exchanges such as NUFAM III.

(1) *Using the Possible Outcomes to Handle Uncertainty in Expected Value Modeling*

A theater-level nuclear exchange may generate many different outcomes which will have a significantly different effect upon any post-nuclear battle. We use the probability models to estimate sets of possible nuclear exchange outcomes, which can be partitioned into sets which we expect to have significantly different effects on the post-exchange conventional battle. The model described in this paper may be used to estimate the probabilities associated with the defeat of each targetable unit, which can be aggregated to determine the probabilities associated with the defeat of each unit represented in FORCEM. If we combine this information with the idea of partitioning of the sample space into outcome sets that lead to significantly different results at the theater level, we can determine the most likely outcome (mode) within each partition, and the joint probability associated with all outcomes within a particular outcome set.

We select an outcome from within each separate partition and use it as input to FORCEM. Each input set represents a different nuclear exchange outcome with (by assumption) significantly different theater-level results. The expected value FORCEM simulation can be run several times, once for each input set, to capture the variability inherent in the nuclear exchange and predict its effect upon the conventional battle. We can associate the probability of an outcome coming from

within a particular partition with the probability of the theater-level result. This provides us with a means for estimating the impact of variability in the input data (in this case, the results of a nuclear exchange) on the output of the expected value model.

The methodology used to partition the outcome sets and select outcomes for input to FORCEM is described fully in CAA-RP-89-5, *Handling Uncertainty in Input to Expected Value Models* (Youngren [1989c]). The methodology developed in this paper permits us to determine all possible exchange outcomes (in terms of the defeat criteria) with their associated likelihoods of occurrence.

(2) *Use as a Research Tool Used to Determine the Sensitivity of Outcomes*

The model described in this paper may also be used as a research tool to determine the sensitivity of various outcomes to different assumptions and sets of input parameters. In theory, this type of analysis can be conducted using any simulation of a theater nuclear exchange such as NUFAM III. In practice, however, major simulation models are sufficiently complex, non-transparent, and time-consuming to set up and run that conducting such analysis becomes lengthy and difficult. The analysis can be performed using part or all of the probability model described in this paper. For example, the impact of using a different warhead may be examined. The effect on preclusion and aggregation can be examined directly by modifying the parameter values in the equations for preclusion and aggregation probabilities; the impact on the probability of defeating a unit can be examined to determine the overall effect.

Because of the assumptions and approximations necessary to yield solutions amenable to calculation in reasonable time, the answers will not be exact. However, the errors in calculations will be less likely to affect conclusions drawn from comparisons and sensitivity studies than they will affect any particular realization. Carrying the distribution forward through the analysis is akin to performing an infinite number of independent samples; errors in closed form estimation of cdf's, etc. may be less than that introduced by small sample sizes resulting from lengthy, costly simulations.

We can also avoid these errors in approximation if we are willing to lose a little of the model transparency. It is possible to estimate the probabilities of preclusion, aggregation, etc. using a statistical Monte Carlo approach (Appendix E). The sensitivity of the results to individual values of the parameters cannot be inferred directly; on the other hand, the accuracy of the results should be more accurate than that obtained in a detailed simulation model using a single realization of an array or using a stylized array.

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(3) *Use as a Surrogate for Detailed Simulation Models such as NUFAM III*

The final use of the model may be as a surrogate to the more detailed simulation models. The advantage of using a model such as this lies in its speed (at least for answers pertaining to individual units) and transparency. The corresponding tradeoff lies in the approximations made to yield calculable results. If the approach of using a Monte Carlo statistical simulation is used to estimate the various location-based probabilities, the answers obtained should be at least as accurate as the comparable answers obtained from a detailed simulation.

If the direct probability calculations are used to determine the probabilities, there will be some error introduced through the simplifying assumptions made in each section. The author intends to conduct further simulation studies to estimate the error of the approximations made herein. The use of the model with direct probability calculations as a surrogate, except in cases where an alternative is not possible, should be considered only after a better appreciation of the source and impact of the errors of approximation are realized.

Section III. MODELING THE ACQUISITION AND MOVEMENT OF TARGET UNITS

Before a potential target unit can be planned for engagement with nuclear weapons, it must be acquired and retained as a viable target long enough to plan the nuclear fires. In order to alter the course of the battle decisively, nuclear weapons use may be constrained to achieve a specific purpose within an appropriate period of time. As a result, weapons may not be fired at potential targets as they are acquired; also, there may be a significant delay between the time the fires are planned and the time of detonation.

Low resolution theater-level models such as FORCEM generally move units periodically using a relatively long time step. In FORCEM, division-sized units locations are updated every 12 hours. Obviously, units subordinate to that division may be in movement during that 12 hour period. This movement may affect our ability to retain the subordinate units as viable targets, and affect our ability to successfully engage them. In order to determine if a unit can be acquired and/or engaged as a stationary target, we must be able to represent the movement of these small units within the 12 hour time step.

It is clear that we must be able to represent the acquisition and movement processes at a higher degree of resolution in time and space then that provided by FORCEM and similar theater-level models or scenarios. Our solution to this problem is to model the acquisition and the movement processes of each unit as independent, alternating renewal processes. This approach is summarized below.

The Target Acquisition Process

Detecting target units with sufficient accuracy to plan for nuclear fires is the process of *target acquisition*. As a consequence of the movement capability of relocatable targets, they do not remain acquired indefinitely (unless they can be tracked indefinitely once acquired); at some time. they move. and the acquisition is no longer valid. Even if a tracking capability exists, there is a probability that such tracking will be lost over time.

The outcome of the target acquisition process is an acquisition list. A target unit is *acquired* when it is detected by a sensor, identified as a target, and placed on the acquisition list. A target unit may be *dropped* from the list either due to a negative sensor report (i.e., we no longer detect its presence), or it may be dropped after some period of time when the acquisition information cannot be updated. Any given target unit will alternate between two states: acquired (retained on the list) or not acquired. A target acquisition process is therefore a temporal series of such acquisition states. The *time to acquisition*, T_a, is the time it takes to acquire a target once any previous acquisition has been dropped; the *time of retention*, T_r, is the time a target is retained on the acquisition list.

Targets that are engaged using conventional weapons are generally fired upon soon after acquisition. Nuclear targets differ from conventional as they are planned for specific purposes dictated by the overall tactical and/or strategic situation. As a result, they are not normally engaged as they are acquired; rather, nuclear fires are directed at targets that are acquired and perceived to be in place at the time the weapons are approved for fire.

Figure 4 illustrates a representative acquisition sequence for a relocatable target unit. Once the unit has been dropped from the list, it is immediately subject to being reacquired. We expect that nuclear weapons use will occur after the conventional battle has been underway for some time; thus. we are interested in the acquisition probabilities at some point in time well after the acquisition process has begun. Our evaluation of the acquisition status of units is made during a short period of

time when planning fires for a particular nuclear exchange; during this short period, we assume that the acquisition situation remains about the same; therefore the times to acquisition $\{T_a\}$ are modeled as independent and identically distributed (iid). The same assumption is made about the times of retention $\{T_r\}$ during this period. For relocatable targets, we can approximate the target acquisition process as an alternating renewal process of indefinite length. Both the time that the target is dropped from the list and the time that the target is acquired are renewal points of this alternating renewal process.
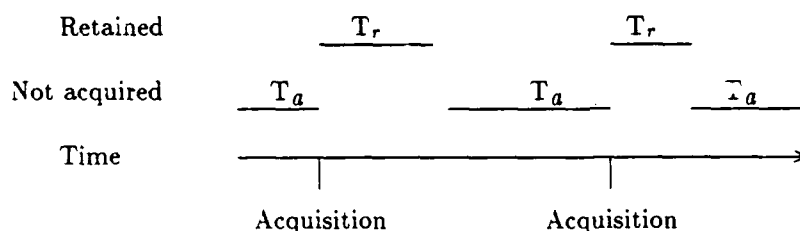
Figure 4. Possible Target Acquisition Sequence

The Movement State of the Relocatable Target

The target unit may be in one of two alternating states with respect to movement: it can be in the *move* state (moving), or it can be in the *stay* state (stationary). We define the random variable S to represent the length of time that a target is stationary and the random variable M to indicate the length of time that it is moving. Again, our evaluation of movement status of units is made during a short period of time when planning fires or firing a particular nuclear exchange; during this short period, we assume that all S and M are mutually independent and are distributed in accordance with distributions F_S and F_M, respectively, and represent the unit movement as an alternating renewal process in the same manner as the target acquisition process.

Representing the Target Acquisition and Movement Process

When constructing a model of the acquisition and engagement of relocatable nuclear targets, it is not necessary to explicitly represent the target acquisition and movement processes in a detailed simulation. Since these processes can be represented as alternating renewal processes, well-known results of renewal theory provide us with the following quantities for any unit:

1. The probability that the unit is currently on an acquisition list.

2. The probability that the unit remains on the acquisition list for any stated interval of time.

3. The probability that the unit was stationary at the time that it was last observed.

4. The probability that the unit remains stationary for any stated interval of time.

From these measures, it is possible to determine the probability (p_{avail}) that a potential target unit is acquired and can be retained as a target until detonation. For additional detail, see Youngren [1989a,b].

Section IV. NOTATION AND TARGET GEOMETRY

Target and Weapon Radii

We assume that target units are circular with a known radius r_{U_i} for unit i. Personnel and equipment are assumed to be uniformly distributed across the unit area, so a defeat criteria of "C percent of casualties to unit i" translates to "C percent of the circular area of unit i covered by casualty-producing nuclear effects." Nuclear effects are also assumed to be circular with a single dominant effect of interest for any particular unit i; we denote the radius of the dominant weapon effect for weapon w as r_W. Note that r_W will also depend upon the defeat criteria; the radius for moderate damage to a particular type of equipment, for example, is larger than the radius for severe damage to the same type of equipment.

The weapon and target geometries discussed in this section are conditioned on knowledge of the location of the unit (target) center and the weapon center. In Chapter 2, we discuss how we remove this conditioning by placing distributions on the unit locations and upon the desired and actual points of weapon detonation.

For a weapon w to be able to achieve a particular known defeat criteria against a specific unit i with known characteristics, the circle of weapon effects r_W drawn around the point of detonation (ground zero) must overlap the circle of radius r_{U_i} drawn around unit i with at least the specified percentage of the target unit area overlapped. For this to occur, the separation between the center of unit i and the ground zero location of unit w must be less than or equal to a distance $d_{i\,w}$. If we consult Figure 5, we see that if we draw a line segment between the point of intersection of the target and weapon circles, we can construct two right triangles. Label the distance between the point of the upper intersection of the target and weapon circles and the line W-U_i as a. The distance from

the center of the weapon circle (ground zero) to the line segment is labeled d_W; the distance from the center of the target unit circle to the line segment is labeled d_{U_i}. The left-hand right triangle has a hypotenuse of length r_{U_i} and sides of length a and d_{U_i}. The right-hand right triangle has a hypotenuse of length r_W and sides of length a and d_W. We denote the total distance from W to U_i as d_{iw}; thus, $d_{iw} = d_W + d_{U_i}$.
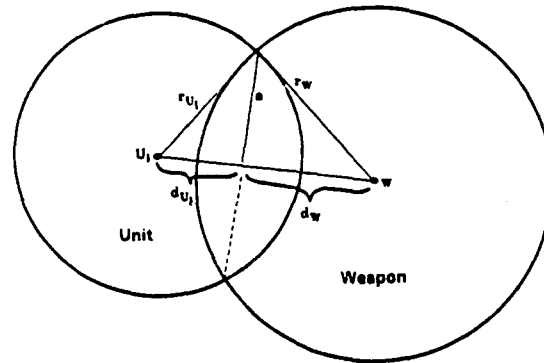
Figure 5. Overlap of Weapon and Unit Circles

Solving for d_W and d_{U_i} as functions of r_{U_i}, r_W, and d_{iw} yields:

$$d_{U_i} = \frac{r_{U_i}^2 - r_W^2 + d_{iw}^2}{2\,d_{iw}} \quad \text{and} \quad d_W = d_{iw} - d_{U_i}.$$

The area of the portion of the intersection on the right (Figure 6) is

$$A_1 = \int_{x=d_{U_i}}^{r_{U_i}} \sqrt{r_{U_i}^2 - x^2} \; dx = \frac{1}{2}\left[\frac{\pi\, r_{U_i}^2}{2} - d_{U_i}\sqrt{r_{U_i}^2 - d_{U_i}^2} - d_{U_i}^2 \sin^{-1}\left(\frac{d_{U_i}}{r_{U_i}}\right) \right].$$
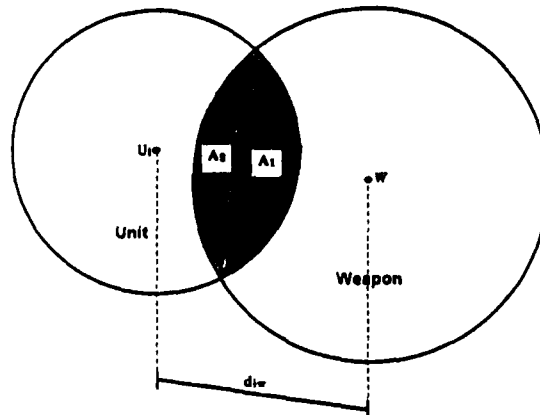
Figure 6. Areas A_1 and A_2 of the Overlap

By symmetry, the area of the portion of the intersection on the left is

$$A_2 = \frac{1}{2}\left[\ \frac{\pi\, r_W^2}{2} - d_W\ \sqrt{r_W^2 - d_W^2}\ - d_W^2\, \sin^{-1}\left(\frac{d_W}{r_W}\right)\ \right].$$

Thus the total area of the intersection is:

$$A_{iw} \equiv A_1 + A_2 = \frac{\pi}{4}\left(r_{U_i}^2 + r_W^2\right)$$

$$- \frac{1}{2}\left[\ d_{U_i}\ \sqrt{r_{U_i}^2 - d_{U_i}^2}\ + d_W\ \sqrt{r_W^2 - d_W^2}\ + d_{U_i}^2\, \sin^{-1}\left(\frac{d_{U_i}}{r_{U_i}}\right) + d_W^2\, \sin^{-1}\left(\frac{d_W}{r_W}\right)\ \right]$$

Since A_{iw} is the area of coverage required to achieve the commander's defeat criteria, and d_{U_i} and d_W can be expressed in terms of r_{U_i}, r_W, and d_{iw}, d_{iw} is the only unknown in this equation. A standard solution technique such as Newton's Method can be used to solve for d_{iw}.

To summarize, we define the following values, which can be computed from input data specifying characteristics of the unit and the weapon :

$r_W \equiv$ Radius of weapon effect (for each specified weapon)

$r_{U_i} \equiv$ radius of unit i.

$d_{iw} =$ the maximum distance that a weapon w can be displaced from the center of mass of a targeted unit i and still achieve the commander's defeat criteria.

$A_{iw} =$ the area of coverage of the target unit i by the weapon effects radius r_W required to achieve the commander's defeat criteria.

Preclusion

Important factors that must be considered in modeling the effects of the use of nuclear weapons are the constraints placed upon DGZ (desired ground zero) placement caused by rules which are intended to preclude damage and injuries to civilian population centers. This is more applicable to US fire planning than Soviet, but the model has to be able to represent the effects for both.

We assume that the preclusion areas are circular. For population centers, these are normally circles that can be drawn around an urban area such that 95% of the population lives within the circle. Large, irregular population centers may consist of more than one overlapping preclusion circle.

We denote the radius of a preclusion area k as r_{P_k}. All weapon DGZ's must be situated such that there is a high degree of confidence that no significant weapon effects will overlap any of the preclusion area. Weapon effects tables (e.g., FM 101-31-2 [1986]) or standard computer programs for weapons effects will provide or compute the necessary offset distance r_W(preclusion), which is a distance at which one can be P% certain (for some confidence level P) that no significant weapon effects will extend. r_W(preclusion) thus includes consideration of the weapon CEP and any appropriate safety factor. Linear preclusion areas (e.g., FLOT for troop safety) may be represented as circles with an outer edge tangent to the point of the linear preclusion area closest to the DGZ. In order to meet preclusion criteria, the DGZ for weapon w can be located no closer than a distance $r_{WP_k} \equiv (\ r_W$(preclusion)$\ +\ r_{P_k}\ )$ to the center of the circular preclusion area k (Figure 7).
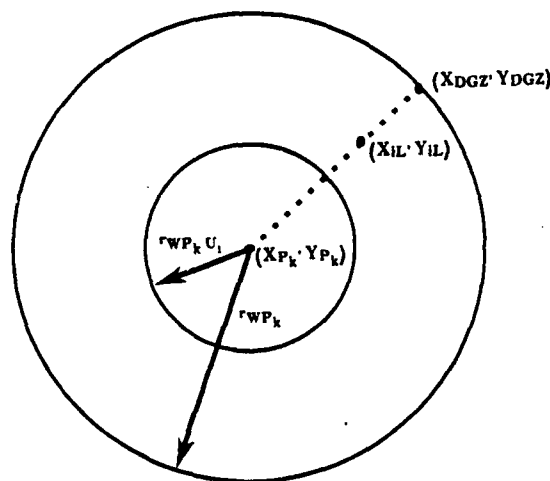
Figure 7. Shift in DGZ Due to Preclusion

If we consider a unit i with radius r_{U_i}, the desired ground zero (for a single target - no aggregation) will always be target center if preclusion considerations do not intervene. If the perceived location of the unit center of mass is at a distance greater than or equal to r_{WP_k} from the preclusion area, then no DGZ shift is necessary. If, on the other hand, the the perceived location of the unit center of mass is at a distance less than r_{WP_k} from the preclusion area, then the DGZ must be shifted in a direction away from the preclusion area. It is possible to displace the DGZ up to a maximum distance of $d_{i\,w}$ and still meet the commander's defeat criteria for that unit. If a shift greater than $d_{i\,w}$ is required, then the unit cannot be engaged with a nuclear weapon of size w. Thus

we define a distance $r_{WP_k U_i}$ as the closest distance that the perceived center of mass location for unit i can approach a preclusion area k using weapon w and still be capable of achieving the defeat criteria for unit i. Clearly, $r_{WP_k U_i} = r_{WP_k} - d_{iw}$ (Figure 8).

In summary, when examining the possibility of shifting a DGZ away from perceived target center to preclude damage to preclusion areas k, we have the following distances defined:

r_W(preclusion) $\equiv$ Radius of weapon effect of concern for preclusion purposes

$r_{U_i} \equiv$ radius of unit i

$r_{P_k} \equiv$ radius of preclusion area k

$r_{WP_k} \equiv r_W$(preclusion) $+ r_{P_k}$, the closest distance that a DGZ for weapon w can approach preclusion area k.

$r_{WP_k U_i} \equiv r_{WP_k} - d_{iw}$, the closest distance that a perceived location for unit i can approach preclusion area k using weapon w and still achieve the defeat criteria for unit i.
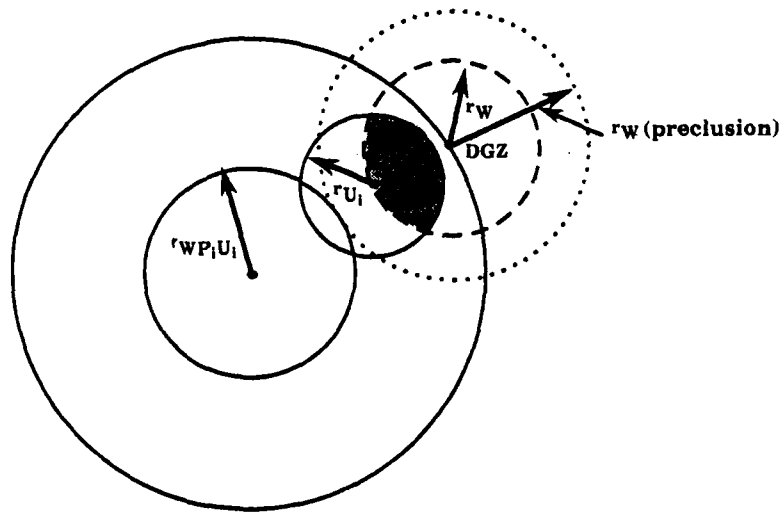
Figure 8. Target Coverage Using a Shifted DGZ

Aggregation

It may be possible to engage two or more target units i, j, ... with a single nuclear weapon, if it is possible to position a weapon w such that it is simultaneously within a distance d_{iw} of unit i, d_{jw} of unit j, etc. In the probability model developed in this paper, we only deal with pairwise aggregation. If we are considering two units i and j for aggregation, we define a distance D_{ij}

between their perceived locations. We refer to the perceived unit locations, rather than the actual locations, because the aggregation is done by target planners based on their knowledge of the units. The perceived unit location of unit i has random coordinates X_{iL} and Y_{iL}; computing the distribution of X_{iL} and Y_{iL} is discussed in Chapter 2. Clearly,

$$D_{ij}^2 = [\, X_{iL} - X_{jL}\,]^2 + [\, Y_{iL} - Y_{jL}\,]^2 \cdot$$

Barring preclusion considerations, the desired ground zero will be located at a point which maximizes the coverage of the weapon effects against both targets. This implies that the DGZ will be located somewhere on a line segment between the perceived locations of units i and j, such that the distance between this point and units i and j is less than d_{iw} and d_{jw} respectively.

Suppose that we are interested in determining the DGZ for an aggregate target formed from units i and j. We will choose the new DGZ along a line segment connecting units i and j (Figure 9). thus we can express the DGZ as a linear combination of the coordinates of units i and j. Let $\alpha\, D_{ij}$ denote the distance from the DGZ to unit j; thus the distance from the DGZ to unit i must be $(1-\alpha)\, D_{ij}$.
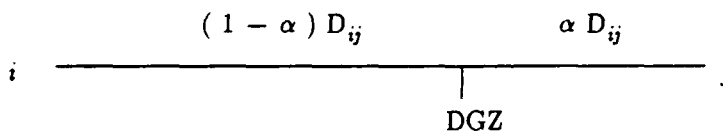
Figure 9. Location of DGZ Given Aggregation (Without Preclusion Shift)

The maximum separation between the two targets, d_{ijw}, is defined as the sum of d_{iw} and d_{jw}. The feasible region for the DGZ lies in the area of overlap of d_{iw} and d_{jw} when $d_{iw} + d_{jw} \geq D_{ij}$ (Figure 10). Clearly $\dfrac{D_{ij} - d_{iw}}{D_{ij}} \leq \alpha \leq \dfrac{d_{jw}}{D_{ij}}$ where α is based on the distance from target j as shown.
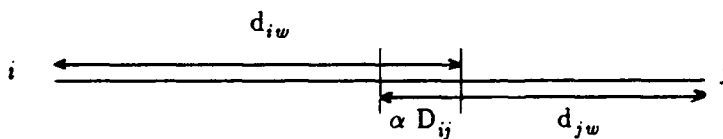
Figure 10. Feasible Area for DGZ Given Aggregation

The random variables representing the coordinates of the DGZ, X_{DGZ} and Y_{DGZ}, are computed from the located (perceived) unit coordinates (X_{iL}, Y_{iL}) and (X_{jL}, Y_{jL}) as follows:

$$X_{DGZ} = \alpha X_{iL} + (1 - \alpha) X_{jL}$$

$$Y_{DGZ} = \alpha Y_{iL} + (1 - \alpha) Y_{jL}$$

The proportion α, $\dfrac{D_{ij} - d_{iw}}{D_{ij}} \leq \alpha \leq \dfrac{d_{jw}}{D_{ij}}$, may be selected in various ways as discussed in Chapter 2.

To recapitulate, the following terms are defined for use in handling aggregation of targets:

$X_{iL} \equiv$ the X-coordinate of the perceived location of unit i

$Y_{iL} \equiv$ the Y-coordinate of the perceived location of unit i

$X_{DGZ} \equiv \alpha X_{iL} + (1 - \alpha) X_{jL}$, the X-coordinate of the aggregate target DGZ

$Y_{DGZ} \equiv \alpha Y_{iL} + (1 - \alpha) Y_{jL}$, the Y-coordinate of the aggregate target DGZ

$d_{ijw} \equiv d_{iw} + d_{jw}$, the maximum separation between the two target units i and j permissible

for the unit to be engaged with weapon w.

Combining Preclusion and Aggregation

The aggregate target formed from a unit pair (i,j) is subject to the same preclusion criteria as any single unit engaged with a nuclear weapon. Thus we check the aggregate DGZ location (with coordinates X_{DGZ} and Y_{DGZ}) to see if the distance from it is less than r_{WP_k} for any preclusion area k, given the weapon w that will be employed against the aggregate target. If the DGZ location is closer than r_{WP_k} to any preclusion area k, the DGZ must be shifted away from the preclusion area. The techniques for computing the DGZ shift are identical to those employed for a single target discussed previously.

Section V. THE UNIT LOCATION MODEL

Introduction

When a subject-matter expert (usually military with field experience) manually arrays small units within a larger sized unit (e.g., battalions within a division), he begins, at least mentally, with a "typical" set of relationships between units based on military doctrine and tactics, given the general scenario. This set of relationships is then modified to take into account terrain features, the

specific scenario-based missions of units, etc. We can picture the "typical" set of relationships as a stylized array or template for the larger unit (e.g., division). Actual unit locations can then be expressed in terms of deviations from the template. As always, when we are *a priori* uncertain about these deviations, we describe our uncertainty in terms of probability distributions.

The Distributional Form

As we saw in Figure 3, we establish cartesian axes such that the y-axis is parallel to the orientation of the unit, and the x-axis is perpendicular to the unit orientation. The model described herein uses the normal distribution to model deviations from each templated unit location in both the x and y directions. That is, let X_i measure the deviation across the width of unit i, and Y_i measure the deviation across the depth of the same unit. The marginal univariate distributions for X_i and Y_i, $i = 1, \ldots, m$, are assumed to be normal.

If we examine actual arrays of units that have been prepared for use in other studies, and plot the contours of likely shifts, the contours tend to be ellipses aligned along one of the two axes. That is, the likely locations for units tend to be parallel to the FLOT for most units; perpendicular to the FLOT for others. Rarely if ever are units primarily oriented diagonally to the FLOT. Thus it is reasonable to assume that all X_i's are mutually independent of all Y_i's for all i, which yields probability contour ellipses parallel or perpendicular to the FLOT.

It is also apparent when examining arrays prepared for other studies that the deviations across the width are not independent, and the deviations across the depth are not independent as well. Because terrain and military tactics tend to cause a similar shift in the width and/or the depth of adjacent units, we desire to establish a model where the the X variables have a positive correlation between adjacent units. The same holds true for the Y variables. We can arrive at such a model if we use a generalized multinormal distribution for the coordinates (X_1, X_2, ..., X_m) and (Y_1, Y_2, ..., Y_m) of m units within a larger unit area. We know that any subset (X_1, X_2, ..., X_s), $s \leq m$, or any set of linear functions of (X_1, X_2, ..., X_m) are also distributed as multinormal. For example, each pair of X variables and pair of Y variables has a bivariate normal distribution with a nonzero correlation. That is, for any adjacent units i and j, the pair (X_i, X_j) $\sim$ BVN[(μ_{xi}, μ_{xj}), $(\sigma_{xi}^2, \sigma_{xj}^2)$, ρ_{xij}] and the pair (Y_i, Y_j) $\sim$ BVN[(μ_{yi}, μ_{yj}), $(\sigma_{yi}^2, \sigma_{yj}^2)$, ρ_{yij}] One of the properties of the bivariate normal distribution is that the variable $U_{ij} \equiv (X_i - X_j) \sim$ N($\mu_{xi} - \mu_{xj}$, $\sigma_{xi}^2 + \sigma_{xj}^2 - 2\rho_{xij}\sigma_{xi}\sigma_{xj}$); $V_{ij} \equiv (Y_i - Y_j)$ is distributed similarly. Thus the

squared distance between unit i and unit j, $(X_i - X_j)^2 + (Y_i - Y_j)^2$, is distributed as the sum of squared normals.

If we consider the variables U_{ij} and U_{jk} which will be used in part to determine the distance between unit i and unit j and the distance between j and k, U_{ij} and U_{jk} are dependent, jointly distributed as bivariate normal when conditioned on X_j.

We will consider aggregation and bonus effects for pairs of targets only. Therefore, we will only be concerned with the distributions of pairs $(X_i - X_j)$ and $(Y_i - Y_j)$ for adjacent units i and j; they will be determined independently and any dependencies between non-adjacent unit pairs will be ignored.

Sources of Data for Array Locations

The prior multinormal distributions for the (X, Y) locations of the units will be obtained from subject matter expert opinion. Any suitable scheme for eliciting expert opinion will suffice; I recommend Lindley [1983] as a good source. The mean vectors will be constructed by asking the subject matter expert(s) to set up a doctrinal template (stylized array) for the forces of interest, given the force structure (dictated by the study) and the overall tactical and operational situation (dictated by the theater-level model or scenario). The separation of units in the template is established on any arbitrary nominal scale; actual coordinates will be generated through some scale multiplier of the overall width and depth of the unit being templated. For example, if templating a division with two brigades on line, each brigade may have a frontage of 1/2 the division frontage; the battalions within the brigades will be similarly separated. Exceptions to this might be the Y-coordinate (distance from FLOT) for artillery units - artillery is generally placed at some constant setback distance from the FLOT regardless of other circumstances. The template (which has mostly relative coordinates) is translated into scaled coordinates given the dimensions of the templated unit. For example, if the division frontage is 10 km, the two forward brigades will have a scaled frontage of 5 km each.

The variance/covariance matrix is similarly established by the subject matter expert on the same nominal scale. Variances will be elicited individually; covariances will be established through independent pairwise comparison. Although this will not yield a true multivariate relationship set, it is generally not possible for experts to generate a large variance/covariance matrix directly (rather

than pairwise). This approach is also consistent with the evaluations of separation distances between units that will be made in Chapter 2. If necessary, a non-informative prior distribution may be used.

The mean vectors, denoted as $\underline{\mu}_x$ and $\underline{\mu}_y$, are assumed to be known for any invariant specified doctrine and theater scenario. The variance/covariance matrices, denoted as Σ_x and Σ_y, are assumed to be unknown and the initial expert-generated matrices will serve as prior distributions on the Σ's. This parameter distribution is updated using standard Bayesian techniques (Appendix C). The multinormal distributions of the unit coordinates will be evaluated conditioned on the current best estimate of ($\underline{\mu}$, Σ).

The sources of the data for updating the matrices Σ are independent realizations of arrays generated manually by subject matter experts on the NUFAM-GAP workstation (see next paragraph, below), starting with the templates. The NUFAM-GAP system allows arrays to be quickly generated with the statistics stored automatically in the workstation.

The NUFAM-GAP Workstation

CAA has acquired a Graphical Analysis Package for the corps-level stochastic nuclear model NUFAM III called NUFAM-GAP. The NUFAM-GAP workstation consists of a PC accompanied by a videodisk player and a TV monitor. Map images of the theater of interest (e.g., central Europe) are displayed from Defense Mapping Agency videodisks onto the TV monitor. The workstation allows for graphic symbols representing unit locations to be superimposed upon the map image. An analyst can use a mouse to easily move the unit symbols on the map, rapidly forming a force array which is captured on a data base.

To use the NUFAM-GAP workstation to generate data for the nuclear force array probability distributions, the analyst starts with a template of the force to be arrayed. This template (which can also be easily created on the workstation) represents the doctrinally most likely positions for the force to be arrayed without reference to terrain. Stylized arrays created for other nuclear studies may be used as one source for array templates. The unit positions on the template represent the mean vector in the X and Y coordinate directions.

After displaying the template on the terrain image, the analyst rapidly shifts the unit positions as necessary to account for the terrain, keeping in mind the scenario and unit mission. The shifts

from the template location are stored in a data base, and the squared distances between shifted locations and template locations represent a single realization of the variance/covariance matrix. The analyst (or other analysts) can do this repetitively, normally using different sections of terrain within the same general area. The realizations form the data used to update the most recent variance/covariance distribution, using Bayes' Law.

Assumptions

Several major assumptions are made in order to produce a model whose results can be computed in reasonable time. These assumptions are summarized below.

a. Independence Between X and Y Coordinates. All X coordinates for all units are assumed to be mutually independent of all Y coordinates for all units.

b. Aggregation and Bonus Effects are Evaluated for Pairs. The probability that any unit j, $j \neq i$, being engaged along with unit i as an aggregate target is considered independently for each j. Only the pairs with the greatest likelihood (largest marginal probability) exceeding some threshold are considered for further evaluation in weapon assignments and probabilities of defeat. Similar considerations are employed for bonus effect calculations. The effect of pairwise consideration forms a bound on the probability that a unit i will be aggregated with some other unit j (see the section on "Evaluating Joint Probabilities" in Chapter 3).

c. Mixing Parameters Approximated as Constants. The coordinates of the Desired Ground Zero (DGZ) shifted due to preclusion and/or aggregation is a linear combination $\alpha X_i + (1-\alpha) X_j$ of some normally-distributed coordinates (X_i, X_j). The mixing parameter α is generally a function of the straight-line distance D_{ij} between the locations of i and j, which makes the linear combination non-normal and lacking a closed-form solution. In order to preserve the normality of the DGZ coordinates and ensure a solution, the mixing parameter α is approximated using $\hat{\alpha}$, a constant which is formed using the expectation of D_{ij}^2. This techniques is employed for both aggregation and preclusion. Preliminary simulation studies show that the approximation yields reasonably accurate results except when the unit mean location is very near a preclusion area, where the approximation understates the probability of engagement.

d. Single Shifts Due to Preclusion. It is possible that the DGZ may be shifted from the ideal location (target center of mass) due to preclusion constraints arising from one or more preclusion areas. We will bound the probability of a DGZ shifting due to preclusion by the probability of the DGZ shift being caused by the preclusion area most likely to cause a shift. Furthermore, we will assume that the shifted DGZ is located on a line through the preclusion area and the prior (unshifted) DGZ location at a distance r_{WP_k} away from the preclusion area (k).

e. Approximate Evaluation of Distributions. The normal cdf lacks a closed-form solution, but many reasonable approximations have been developed which are used herein. The distribution for quadratic forms in Normal variables must also be approximated (see Appendix D).

f. Targeting Assumptions. There are many assumptions (standard in nuclear effects models) relating to the nuclear weapon and the target. The target location error and the Circular Error Probable (CEP) of the weapon are assumed to be distributed as circular Normal. The unit areas. preclusion areas and the weapon effect areas are circular, with target elements distributed uniformly over the target unit area. We also assume that at most one weapon will be employed against any one single target; if this is not true *a priori*, the data base needs to be defined using multiple targets located at the same point (without bonus effects).

g. Independence of Evaluation of Results between Units. In order to evaluate the probability that a unit i is defeated, we assume that the probability of defeating unit i (with any weapon) is independent of the probability of defeating unit j $\forall$ $j \neq i$.

h. The Center-of-Mass for each Unit Templated is Known. When we form a template for a large unit, say a division, we assume that the division center-of-mass relative to some terrain coordinates is known.

CHAPTER 2

DETERMINING THE DISTRIBUTION OF THE DESIRED GROUND ZERO

Section I. CHAPTER SUMMARY

Summary of Contents

In this chapter, we will determine the various ways that a Desired Ground Zero (DGZ) may be established. For each way, we will also determine the appropriate distribution for the DGZ. The DGZ distribution is the distribution of the aimpoint of the weapon. When we include consideration of the weapon accuracy and reliability, we can use the DGZ distribution to determine the distribution of the point of detonation (if a detonation occurs) - the Actual Ground Zero (AGZ) location.

If a target unit i is available (has been acquired and will be retained through the time of detonation, as discussed in Section III of the previous chapter), then it may be engaged in the following ways with a particular weapon type w:

(1) Engaged as a single target with no DGZ shift. This will occur if there is no suitable target unit j, $j \neq i$, available for aggregation, unit i is not precluded by a preclusion area, and all preclusion areas are sufficiently far away that the DGZ need not be shifted away from them. In this case, the DGZ will be located at the center of the perceived target location.

(2) Engaged as a single target with a DGZ shift. This will occur if there is no suitable target unit j, $j \neq i$, available for aggregation, unit i is not precluded by a preclusion area, but there is at least one preclusion area sufficiently close that the DGZ needs to be shifted away from it. In this case, the DGZ will be located at a point opposite the preclusion area on a line drawn from the closest preclusion area through the center of the perceived target location.

(3) Engaged as part of an aggregate target with no DGZ shift. This will occur if there is at least one suitable target unit j, $j \neq i$, available for aggregation, unit i is not precluded by a preclusion area, and all preclusion areas are sufficiently far away from the aggregate target DGZ that the DGZ

need not be shifted away from them. In this case, the DGZ will be located at a point in between the two target units on a line drawn between the center of the perceived target location and the center of the location of the closest target unit j.

(4) Engaged as part of an aggregate target with a DGZ shift. This will occur if there is at least one suitable target unit j, $j \neq i$, available for aggregation, unit i is not precluded by a preclusion area, but there is at least one preclusion area sufficiently close that the DGZ needs to be shifted away from it. In this case, the DGZ will be located at a point opposite the preclusion area on a line drawn from the closest preclusion area through the unshifted aggregate DGZ (the DGZ that would have been selected in paragraph (3) above).

(5) Unit i is not engaged. This will occur if unit i is precluded by any preclusion area.

This determination is made for all weapons w. Note that it is possible for different outcomes to occur with different weapons. If we are doing a Monte Carlo realization of the model, then only one of the five outcomes above will occur for each weapon system for each replication. If we are solving the model analytically, then there is a probability associated with each outcome. For example, for a particular i and w, outcome 1 may occur with probability 0.2, outcome 2 with probability 0.35, outcome 3 with probability 0.1, outcome 4 with probability 0.2, and outcome 5 with probability 0.15. In this case, the requirement is simply that all out the outcome *probabilities* must sum to 1.

Summary of the Logic

The logic followed in this determination is shown in Figure 11. The probabilities listed next to each line will be explained in detail later in the text. Beginning in the lower left hand corner, the following steps occur for a each unit i and weapon w:

(1) The distribution of the actual unit location (X_i, Y_i) is determined using the template information (which yields the mean location) and the other distributional information (variance/covariance vector).

(2) The acquisition and movement models are used to determine the probability that unit i is available. The probability that unit j is available is also computed for all $j \neq i$.
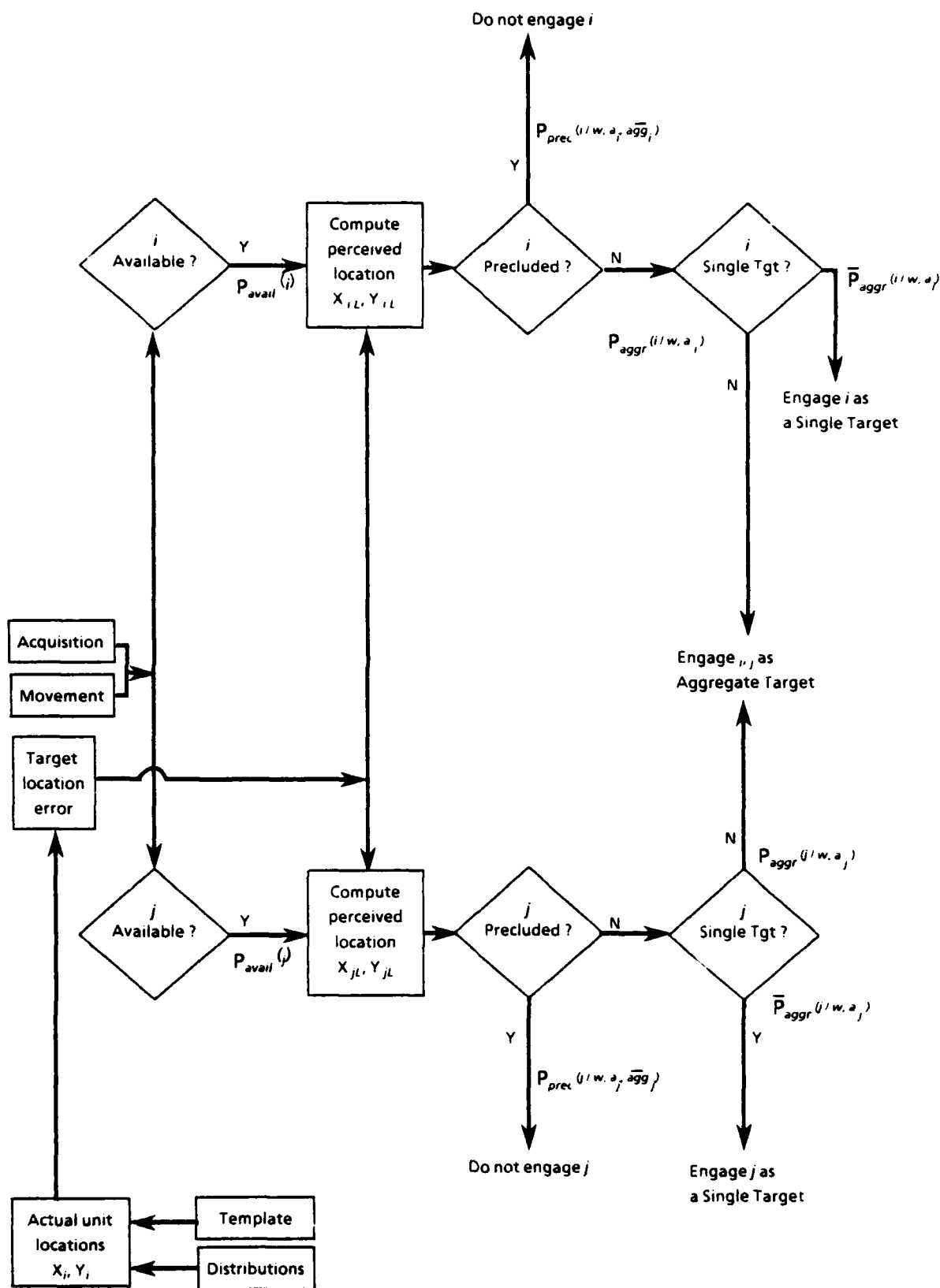
Figure 11. Logic for Determining the Engagement Type for Unit i

(3) The perceived unit location (X_{iL}, Y_{iL}) is derived for all available units by combining the distribution of the target location error with the distribution of the actual unit locations.

(4) The distribution of the distance between the perceived unit location of unit i and all preclusion areas is computed. The distribution of the distance to the nearest preclusion area (in terms of probability) determines the probability that unit i is precluded. This calculation is also made for units j that may be aggregated with unit i.

(5) The distribution of the distance between unit i and all units j, $j \neq i$, is made to determine the closest (in terms of probability - that is, the unit j that has the highest probability of being close enough to aggregate using weapon w).
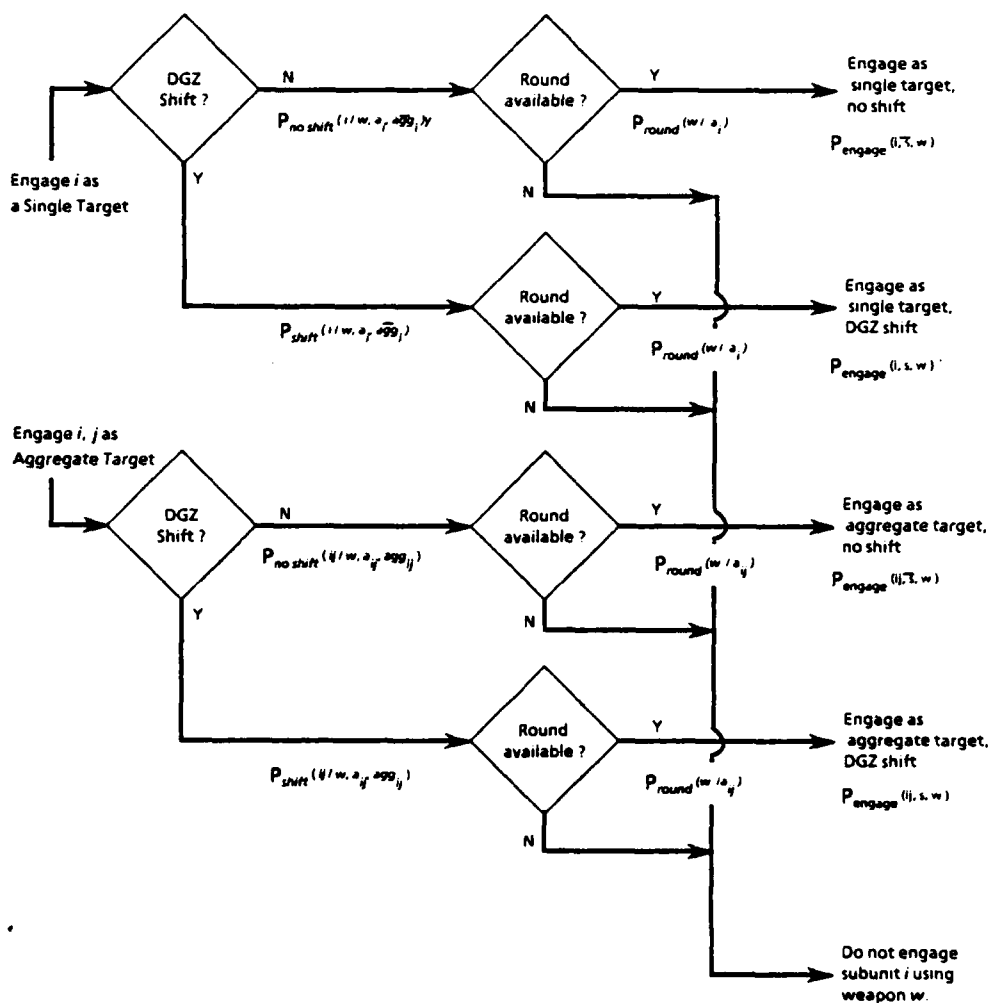

Figure 12. Logic for Determining the Engagement Type for Unit i (Continued)

At this point, there are 3 possibilities with respect to unit i: It may be engaged as a single target, as an aggregate target with the closest unit j, or it may not be engaged at all. Figure 12 continues this logic by looking at the possibility of having to shift a DGZ away from the preferred location (closest to unit center(s)) due to preclusion.

(6) For i engaged as a single target, the distribution of the distance between the perceived unit location of unit i and closest preclusion area is used to determine the distribution of the DGZ if a DGZ shift is required. Otherwise, the DGZ (unshifted) is located at the perceived center of unit i.

(7) For i engaged as an aggregate target, the distribution of the distance between the unshifted DGZ (for unit i and the closest perceived unit j) and the closest preclusion area is used to determine the distribution of the DGZ if a DGZ shift is required. Otherwise, the DGZ (unshifted) is located between the perceived centers of units i and j.

(8) For each engagement possibility with unit i and weapon type w, there is an associated probability p_{round} that a round of weapon type w will be available for use to engage unit i. The detail on how to compute these probabilities is found in Chapter 3.

The remainder of this chapter derives the distributions for these DGZs.

Summary of the Results

If a target unit i is available (has been acquired and will be retained through the time of detonation, as discussed in Section III of the previous chapter), then the distributions of the DGZ for each way that it may be engaged with a particular weapon type w follow:

(1) Engaged as a single target with no DGZ shift.
$$X_{DGZ} = X_{iL} \sim N(\ \mu_{zi} + \mu_{tzi},\ \sigma_{zi}^2 + \sigma_{tzi}^2\ )$$
$$Y_{DGZ} = Y_{iL} \sim N(\ \mu_{yi} + \mu_{tyi},\ \sigma_{yi}^2 + \sigma_{tyi}^2\ )$$

(2) Engaged as a single target with a DGZ shift (the superscript "s" indicates a shifted DGZ).
$$X_{DGZ}^s = \beta\ X_{iL} + (\ 1 - \beta\ )\ x_{P_k} \sim N(\ \beta(\mu_{zi} + \mu_{tzi}) - (1-\beta)x_{P_k},\ \beta^2(\sigma_{zi}^2 + \sigma_{tzi}^2)\ )$$
$$Y_{DGZ}^s = \beta\ Y_{iL} + (\ 1 - \beta\ )\ y_{P_k} \sim N(\ \beta(\mu_{yi} + \mu_{tyi}) - (1-\beta)y_{P_k},\ \beta^2(\sigma_{yi}^2 + \sigma_{tyi}^2)\ )$$

(3) Engaged as part of an aggregate target with no DGZ shift.

$$X_{DGZ} = \alpha X_{iL} + (1 - \alpha) X_{jL}$$
$$\sim N\Big(\alpha(\mu_{xi} + \mu_{txi}) + (1-\alpha)(\mu_{xj} + \mu_{txj}),$$
$$\alpha^2(\sigma_{xi}^2 + \sigma_{txi}^2) + (1-\alpha)^2(\sigma_{xi}^2 + \sigma_{txi}^2) + 2\alpha(1-\alpha)\rho_{xij}\sigma_{xi}\sigma_{xj} \Big)$$

$$Y_{DGZ} = \alpha Y_{iL} + (1 - \alpha) Y_{jL}$$
$$\sim N\Big(\alpha(\mu_{yi} + \mu_{tyi}) + (1-\alpha)(\mu_{yj} + \mu_{tyj}),$$
$$\alpha^2(\sigma_{yi}^2 + \sigma_{tyi}^2) + (1-\alpha)^2(\sigma_{yi}^2 + \sigma_{tyi}^2) + 2\alpha(1-\alpha)\rho_{xij}\sigma_{yi}\sigma_{yj} \Big)$$

(4) Engaged as part of an aggregate target with a DGZ shift.

$$X^s_{DGZ} = \beta X_{DGZ} + (1-\beta)x_{P_k} = \beta[\alpha X_{iL} + (1-\alpha)X_{jL}] + (1-\beta)x_{P_k}$$
$$\sim N\Big(\alpha\beta(\mu_{xi} + \mu_{txi}) + (1-\alpha)\beta(\mu_{xj} + \mu_{txj}) - (1-\beta)x_{P_k},$$
$$(\alpha\beta)^2(\sigma_{xi}^2 + \sigma_{txi}^2) + ((1-\alpha)\beta)^2(\sigma_{xi}^2 + \sigma_{txi}^2) + 2\alpha(1-\alpha)\beta^2\rho_{xij}\sigma_{xi}\sigma_{xj} \Big)$$

$$Y^s_{DGZ} = \beta Y_{DGZ} + (1-\beta)y_{P_k} = \beta[\alpha Y_{iL} + (1-\alpha)Y_{jL}] + (1-\beta)y_{P_k}$$
$$\sim N\Big(\alpha\beta(\mu_{yi} + \mu_{tyi}) + (1-\alpha)\beta(\mu_{yj} + \mu_{tyj}) - (1-\beta)y_{P_k},$$
$$(\alpha\beta)^2(\sigma_{yi}^2 + \sigma_{tyi}^2) + ((1-\alpha)\beta)^2(\sigma_{yi}^2 + \sigma_{tyi}^2) + 2\alpha(1-\alpha)\beta^2\rho_{yij}\sigma_{yi}\sigma_{yj} \Big)$$

(5) Unit i is not engaged. In this case, there is no DGZ.

The symbols used in the above expressions are:

μ_{xi}, μ_{yi}: the mean actual locations (in the x and y directions) of unit i as defined in the templates.

σ_{xi}^2, σ_{yi}^2: the variance of the actual locations of unit i in the x and y directions.

μ_{txi}, μ_{tyxi}, σ_{txi}^2, σ_{tyi}^2: mean and variances of the shifts associated with the target location errors of unit i.

x_{P_k}, y_{P_k}: The coordinates of preclusion area k (as used, the closest preclusion area).

X_{iL}, Y_{iL}: The coordinates of the perceived location of unit i.

α: The mixing parameter which determines the location of the unshifted aggregate DGZ.

β: The mixing parameter which determines the location of a shifted DGZ.

The remainder of this chapter simply presents the mathematics behind the distributions for these DGZs.

Section II. DERIVATIONS AND RESULTS

The Marginal Distribution of the Actual Unit Locations

The actual unit location for a unit i is expressed in terms of the X-coordinate, X_i, and the Y-coordinate, Y_i. The joint distribution for all m X-coordinates X_1, ..., X_m is distributed as multinormal(μ_x, Σ_x) and the joint distribution for all m Y-coordinates Y_1, ..., Y_m is distributed as multinormal(μ_y, Σ_y). The vector μ is known (derived from templates) and the matrix Σ is distributed as Wishart (Appendix C). For all of the calculations to follow, Σ is assumed to take on its mean value. The marginal distribution of any X_i, $i = 1$, ..., m is distributed as Normal(μ_{xi}, σ_{xi}^2) and the marginal distribution of any Y_i, $i = 1$, ..., m is distributed as Normal(μ_{yi}, σ_{yi}^2).

Determining the Marginal Distribution of the Perceived Unit Location

When a unit is acquired, there is a possible target location error (TLE) associated with the acquisition. The TLE is modeled using a bivariate normal distribution (in the X and Y direction) from the aimpoint with zero correlation. Since the marginal distributions of X_i and Y_i are normally distributed for any unit i, the perceived unit location associated with that unit is distributed as the sum of two normals. Let TX_i and TY_i denote the target location error in the x and y direction, respectively, for unit i and let X_{iL} and Y_{iL} denote the located (perceived) coordinates of unit i. Then

$$X_{iL} = (X_i + TX_i) \sim N(\mu_{xi} + \mu_{txi}, \sigma_{xi}^2 + \sigma_{txi}^2)$$
$$Y_{iL} = (Y_i + TY_i) \sim N(\mu_{yi} + \mu_{tyi}, \sigma_{yi}^2 + \sigma_{tyi}^2)$$

Determining the Distribution of Distance Between Targets

In order to determine if it is possible to engage two targets with one weapon, one must determine the distance between the two. We assume that all aggregation will occur only between pairs of closest adjacent units. The determination will be made by the target analyst based on the unit aimpoints (perceived center of mass if no shifts occur due to preclusion), not actual unit locations. Using our normal model given above, the squared distance between two adjacent units i and j, denoted as D_{ij}^2, is:

$$D_{ij}^2 = [(X_i + TX_i) - (X_j + TX_j)]^2 + [(Y_i + TY_i) - (Y_j + TY_j)]^2 .$$

If we rearrange the terms inside the square brackets, we see that this is the sum of two squared normals.

$$D_{ij}^2 = [X_i - X_j + TX_i - TX_j]^2 + [Y_i - Y_j + TY_i - TY_j]^2,$$

where

$$D_{xij} \equiv X_i - X_j + TX_i - TX_j \sim N(\mu_{xij}, \sigma_{xij}^2)$$

$$D_{yij} \equiv Y_i - Y_j + TY_i - TY_j \sim N(\mu_{yij}, \sigma_{yij}^2)$$

$$\mu_{xij} \equiv (\mu_{xi} + \mu_{txi} - \mu_{xj} - \mu_{txj})$$

$$\sigma_{xij}^2 \equiv (\sigma_{xi}^2 + \sigma_{txi}^2 + \sigma_{xj}^2 + \sigma_{txj}^2 - 2\rho_{xij}\sigma_{xi}\sigma_{xj})$$

$$\mu_{yij} \equiv (\mu_{yi} + \mu_{tyi} - \mu_{yj} - \mu_{tyj})$$

$$\sigma_{yij}^2 \equiv (\sigma_{yi}^2 + \sigma_{tyi}^2 + \sigma_{yj}^2 + \sigma_{tyj}^2 - 2\rho_{yij}\sigma_{yi}\sigma_{yj}).$$

To evaluate these terms, let us define unit normal variables Z_{xij} and Z_{yij} such that:

$$Z_{xij} \equiv \frac{(X_i - X_j + TX_i - TX_j) - \mu_{xij}}{\sigma_{xij}} \sim N(0,1) \qquad \text{and}$$

$$Z_{yij} \equiv \frac{(Y_i - Y_j + TY_i - TY_j) - \mu_{yij}}{\sigma_{yij}} \sim N(0,1).$$

Then $D_{ij}^2 = D_{xij}^2 + D_{yij}^2 = \sigma_{xij}^2 (Z_{xij} + \frac{\mu_{xij}}{\sigma_{xij}})^2 + \sigma_{yij}^2 (Z_{yij} + \frac{\mu_{yij}}{\sigma_{yij}})^2$

and we know that $(Z_{xij} + \frac{\mu_{xij}}{\sigma_{xij}})^2 \sim \chi_1^2(\lambda_x)$ and $(Z_{yij} + \frac{\mu_{yij}}{\sigma_{yij}})^2 \sim \chi_1^2(\lambda_y)$,

where $\chi_\nu^2(\lambda)$ denotes a non-central chi-square distribution with degrees of freedom ν and a non-centrality parameter λ.

If we define the general quadratic form $Q(\underline{Z}) \equiv \sum_{j=1}^{n} \lambda_j (Z_j - \mu_{zj})^2$, where each $Z_j \sim N(0,1)$, then we see that the squared distance between units i and j, D_{ij}^2, is a quadratic form in normal variables. For evaluating the distribution of D_{ij}^2, see Appendix D. In most cases, the distribution of D_{ij}^2 can be approximated as Normal $\left[\sum_{k=x}^{y} \sigma_{kij} \mu_{kij}, \sum_{k=x}^{y} 2 (1 + 2 \frac{\mu_{kij}}{\sigma_{kij}}) \sigma_{kij}^4 \right]$.

Modeling Aggregated Target DGZs

The effect of target aggregation is to shift the DGZ for a detonation from target center (or the point closest to target center allowed by preclusion considerations) to a point along a line drawn between the centers of two targets that will be at least partially covered by the effects of a single weapon. Aggregation can occur between units i and j using weapon w whenever the perceived centers

of i and j are within some maximum allowable distance d_{ijw} of each other, where d_{ijw} is determined by the weapon effects radii of weapon w, the sizes of targets i and j, and the commander's guidance that dictates the minimum acceptable coverage of targets i and j by weapon w. Implicit in this discussion is the assumption that both targets i and j have been acquired, otherwise aggregation cannot occur (although bonus effects might).

Suppose that we are interested in determining the DGZ for an aggregate target formed from units i and j. Recall that, barring preclusion considerations, we will choose the new DGZ along a line segment connecting units i and j with coordinates as follows:

$$X_{DGZ} = \alpha\, X_{iL} + (1 - \alpha)\, X_{jL}$$
$$Y_{DGZ} = \alpha\, Y_{iL} + (1 - \alpha)\, Y_{jL}$$

The proportion α, $0 \leq \alpha \leq 1$, may be selected in either one of three different ways:

a. Weighted by Target Priority:

(1) For targets of equal priority and size, $\alpha = \dfrac{D_{ij} - d_{iw}}{D_{ij}} + \dfrac{1}{2}\left[\dfrac{d_{jw}}{D_{ij}} - \dfrac{D_{ij} - d_{iw}}{D_{ij}}\right]$

(2) For targets of equal priority but different size, α is weighted by size. Suppose r_{U_j}, the radius of target j, is greater than r_{U_i}. Then the DGZ should be closer to target j than i. To do this, we note that $\dfrac{r_{U_i}}{r_{U_i} + r_{U_j}} < \dfrac{1}{2}$ so we set $\alpha = \dfrac{D_{ij} - d_{iw}}{D_{ij}} + \dfrac{r_{U_i}}{r_{U_i} + r_{U_j}}\left[\dfrac{d_{jw}}{D_{ij}} - \dfrac{D_{ij} - d_{iw}}{D_{ij}}\right]$

(3) For targets of different priority and potentially different size, we want to place the DGZ closer to the higher priority target, taking into account variations in size. We propose a simple multiplier to the α computed using (1) or (2) above, based on the priority numbers given to each type target, with priority 1 indicating the highest priority, 2 the second highest priority, etc. If target i has priority $[i]$ and target j has priority $[j]$, then we let α equal the smallest value of $\dfrac{d_{jw}}{D_{ij}}$ and $\dfrac{D_{ij} - d_{iw}}{D_{ij}} + \dfrac{r_{U_i}}{r_{U_i} + r_{U_j}}\left[\dfrac{d_{jw}}{D_{ij}} - \dfrac{D_{ij} - d_{iw}}{D_{ij}}\right] \cdot \left[1 + \dfrac{[j] - [i]}{\max\{[i], [j]\}}\right]$. If $[j] \leq [i]$ (thus target j is more important), then α will be smaller and the DGZ will be closer to target j. The opposite will be true if $[j] > [i]$; if $[j] = [i]$, α remains unchanged.

b. DGZ Set as Close as Possible to Target of Greatest Priority.
In this case, we set α such that the DGZ is as close to the target of greatest priority as possible while still achieving defeat

criteria against the secondary target. To simplify exposition, suppose that $[j] \leq [i]$ (thus j is more important than i). Then we want α to be as small as possible and the DGZ to be at a distance $D_{ij} - d_{iw}$ from target j. This is achieved when $\alpha = \dfrac{D_{ij} - d_{iw}}{D_{ij}}$ regardless of any other consideration.

c. α **Set to be a Constant.** If we desire a constant α (rather than one dependent upon D_{ij}), we can use any of the formulas in a. or b. above and substitute $\hat{D}_{ij} \equiv$ $\sqrt{ E[\, D_{ij}^2 \mid 0 \leq D_{ij}^2 \leq (d_{iw} + d_{iw})^2 \,] }$, the conditional expectation of D_{ij}, for D_{ij} (the formula for computing this conditional expectation is given in the section on preclusion given aggregation). The main reason for doing this is that the random variables representing the coordinates of the DGZ, X_{DGZ} and Y_{DGZ}, remain independent and normally distributed if α is constant. If α is a random variable (a function of D_{ij}), then we lack a closed-form expression for the distributions of X_{DGZ} and Y_{DGZ}.

Modeling Preclusion Given No Aggregation

We condition the results in this section on the event $D_{ij}^2 > (d_{iw} + d_{jw})^2 = d_{ij\,w}^2$; that is, on the event that the units i and j are sufficiently far apart that it is not possible to achieve the defeat criteria against both targets using a *single weapon of type w*. As a result, we are interested in examining possible shifts due to preclusion constraints as applied against each target i and j individually. Barring any preclusion constraints, the DGZ of the weapon would be at the located (perceived) unit center of mass; that is, at the coordinates (X_{iL}, Y_{iL}) for unit i.

For each target i, let D_{iP_k} denote the distance between preclusion area k and target unit i. Recall that $r_{WP_k} \equiv r_W(\text{preclusion}) + r_{P_k}$, the closest distance that a DGZ for weapon w can approach preclusion area k, and $r_{WP_k U_i} \equiv r_{WP_k} - d_{iw}$, the closest distance that a perceived location for unit i can approach preclusion area k using weapon w and still achieve the defeat criteria. With regard to the target unit i, preclusion area k and weapon w, there are several possible outcomes:

a. $D_{iP_k}^2 < r_{WP_k U_i}^2$. If this outcome occurs, target i cannot be engaged with weapon w.

b. $D_{iP_k}^2 \geq r_{WP_k}^2$ for all preclusion areas. If this outcome occurs, target i can be engaged with weapon w without a shift in DGZ caused by preclusion.

c. $r^2_{WP_k} > D^2_{iP_k} \geq r^2_{WP_k U_i}$ for some preclusion area k. If this outcome occurs, target i can be engaged with weapon w, but the DGZ will need to be shifted away from the located (perceived) target center of mass to a distance no closer than r_{WP_k} to the center of mass of preclusion area k. In order to achieve as much coverage of the target as possible, we assume that the DGZ will be shifted along a line drawn through the center of mass of preclusion area k and the perceived center of mass of target unit i to a point a distance of exactly r_{WP_k} away from the center of the preclusion area (Figure 13). Thus the coordinates of the shifted DGZ, X^s_{DGZ} and Y^s_{DGZ}, solve the equation $(X^s_{DGZ} - X_{P_k})^2 + (Y^s_{DGZ} - Y_{P_k})^2 = r^2_{WP_k}$, where (X_{P_k}, Y_{P_k}) denote the coordinates of the preclusion area k center of mass. If we define a random variable β such that

$$\beta^2 = \frac{r^2_{WP_k}}{D^2_{iP_k}}, \text{ then } X^s_{DGZ} = \beta X_{iL} + (1 - \beta) X_{P_k} \text{ and } Y^s_{DGZ} = \beta Y_{iL} + (1 - \beta) Y_{P_k}.$$
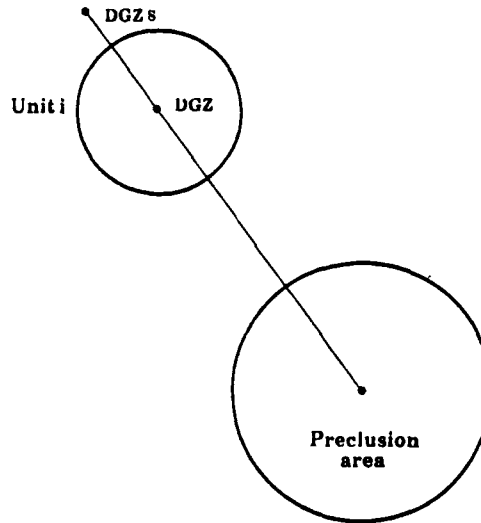

Figure 13. Shift in the DGZ Due to Preclusion (Single Target)

We approximate β using the constant $\dfrac{r_{WP_k}}{\sqrt{E[D^2_{iP_k}]}}$, which allows X^s_{DGZ} and Y^s_{DGZ} to remain independent and normally distributed. $E[D^2_{iP_k}]$ is computed from the identity $D^2_{iP_k} = (X_{iL} - X_{P_k})^2 + (Y_{iL} - Y_{P_k})^2$ for each target i, ignoring (as an approximation) the dependence on $D^2_{ij} > d_{ijw}$ in the evaluation of the expectation. For information on how joint dependencies are approximated in this paper, see the section on "Evaluating Joint Probabilities" in Chapter 3.

Modeling Preclusion Given Aggregation

We condition the results in this section on the event $D_{ij}^2 \leq (d_{iw} + d_{jw})^2$; that is, on the event that the units i and j are sufficiently close that it is possible to achieve the defeat criteria against both targets using a single weapon of type w. As a result, we are interested in examining possible shifts in DGZ due to preclusion constraints as applied against the DGZ of the aggregated targets. Barring any preclusion constraints, the DGZ of a weapon used against targets i and j would be along a line segment connecting i and j; that is, at the coordinates $X_{DGZ} = \alpha X_{iL} + (1 - \alpha) X_{jL}$. $Y_{DGZ} = \alpha Y_{iL} + (1 - \alpha) Y_{jL}$, where α is determined as stated in the section on aggregation.

For the aggregate target formed from units i and j, let $D_{ij P_k}$ denote the distance between preclusion area k and aggregate target ij. If we assume that a shifted DGZ will be placed along a line drawn through the center of mass of preclusion area k and the previous (unshifted) DGZ located at coordinates (X_{DGZ}, Y_{DGZ}) to a point a distance of exactly r_{WP_k} away from the center of the preclusion area (Figure 14), the coordinates of the shifted DGZ, X_{DGZ}^s and Y_{DGZ}^s, solve the equation $(X_{DGZ}^s - X_{P_k})^2 + (Y_{DGZ}^s - Y_{P_k})^2 = r_{WP_k}^2$.
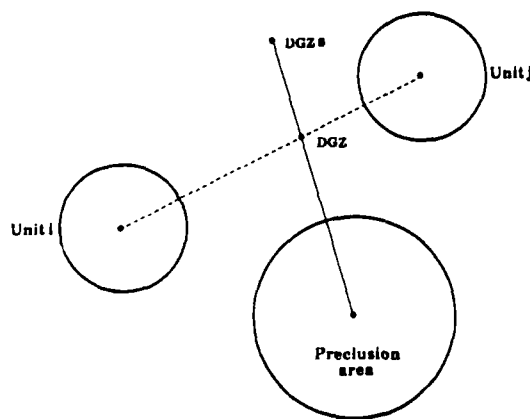

Figure 14. Shift in the DGZ Due to Preclusion (Aggregate Target)

Let us define a distance between the aggregate DGZ and preclusion area k as $D_{D P_k}^2 = (X_{DGZ} - X_{P_k})^2 + (Y_{DGZ} - Y_{P_k})^2$. Once again there are several possible outcomes:

a. $D_{D P_k}^2 < r_{WP_k U_{ij}}^2$ for some preclusion area k, where $r_{WP_k U_{ij}} \equiv \max\{r_{WP_k U_i}, r_{WP_k U_j}\}$. If this outcome occurs, either target i or target j cannot be engaged with weapon w at the aggregate DGZ (one or both targets may still be able to be engaged as single targets).

b. $D^2_{DP_k} \geq r^2_{WP_k}$ for all preclusion areas. If this outcome occurs, the aggregate target formed from units i and j can be engaged with weapon w without a shift in DGZ caused by preclusion.

c. $D^2_{DP_k} \geq r^2_{WP_k U_{ij}}$ for all preclusion areas, but $D^2_{DP_k} < r^2_{WP_k}$ for some k, and
$$(X^s_{DGZ} - X_{iL})^2 + (Y^s_{DGZ} - Y_{iL})^2 > d^2_{iw} \text{ and/or}$$
$$(X^s_{DGZ} - X_{jL})^2 + (Y^s_{DGZ} - Y_{jL})^2 > d^2_{jw}.$$

Because $D^2_{DP_k} < r^2_{WP_k}$, the DGZ will need to be shifted away from the aggregate DGZ to a distance no closer than r_{WP_k} to the center of mass of preclusion area k at coordinates (X^s_{DGZ}, Y^s_{DGZ}) which solve the equation $(X^s_{DGZ} - X_{P_k})^2 + (Y^s_{DGZ} - Y_{P_k})^2 = r^2_{WP_k}$. If we define random variable β such that $\beta^2 = \dfrac{r^2_{WP_k}}{D^2_{DP_k}}$, then

$$X^s_{DGZ} = \beta X_{DGZ} + (1 - \beta) X_{P_k} \text{ and } Y^s_{DGZ} = \beta Y_{DGZ} + (1 - \beta) Y_{P_k}.$$

Once again, we are unable to obtain a closed form solution to the distributions of X^s_{DGZ} and Y^s_{DGZ} unless we approximate β as a constant. We let

$$\beta = \frac{r_{WP_k}}{\sqrt{E[D^2_{DP_k} | D^2_{ij} \leq (d_{iw} + d_{jw})^2 \cap r^2_{WP_k U_{ij}} \leq D^2_{DP_k} < r^2_{WP_k}]}},$$ which allows X^s_{DGZ} and Y^s_{DGZ} to

remain independent and normally distributed. In this case, we condition the expectation of D_{DP_k} on the event that $D^2_{ij} \leq (d_{iw} + d_{jw})^2$; that is, on the event that the units i and j are sufficiently close that it is possible to achieve the defeat criteria against both targets using a single weapon of type w. We make this correction in this case (preclusion with aggregation) since the unshifted DGZ location is based on a linear function of D^2_{ij}.

We can solve for $E[D^2_{DP_k} | D^2_{ij} \leq (d_{iw} + d_{jw})^2]$ by noting that if θ denotes the angle between the line segment connecting units i and j and the line segment connecting unit j with preclusion area k, then

$$D^2_{DP_k} = \alpha^2 D^2_{ij} + D^2_{jP_k} - 2 \alpha D_{ij} D_{jP_k} \cos\theta \text{ and } D^2_{iP_k} = D^2_{ij} + D^2_{jP_k} - 2 D_{ij} D_{jP_k} \cos\theta.$$
Thus

$$D^2_{DP_k} =(\alpha^2 - \alpha) D^2_{ij} + (1 - \alpha) D^2_{jP_k} + \alpha D^2_{iP_k}.$$

Also, $r^2_{WP_k U_{ij}} \leq D^2_{DP_k} < r^2_{WP_k}$ is therefore equivalent to

$$\alpha r^2_{WP_k U_i} + (1-\alpha) r^2_{WP_k U_j} \leq \alpha D^2_{iP_k} + (1-\alpha) D^2_{jP_k} < r^2_{WP_k} + (\alpha - \alpha^2)(d_{iw} + d_{jw})^2$$

since $\alpha D^2_{iP_k} + (1-\alpha) D^2_{jP_k} = D^2_{DP_k} - (\alpha^2 - \alpha) D^2_{ij}$ and $r^2_{WP_k U_{ij}} \leq D^2_{DP_k}$ implies both $r^2_{WP_k U_i} \leq D^2_{iP_k}$ and $r^2_{WP_k U_j} \leq D^2_{jP_k}$.

Continuing our approximation that D_{iP_k}, D_{jP_k} and D_{ij} are independent,

$$E[\, D^2_{DP_k} \mid D^2_{ij} \leq (\, d_{iw} + d_{jw}\,)^2] \doteq (\, \alpha^2 - \alpha\,)\, E[\, D^2_{ij} \mid D^2_{ij} \leq (\, d_{iw} + d_{jw}\,)^2]$$

$$+ E[\![\, \alpha D^2_{iP_k} + (1-\alpha)\, D^2_{jP_k} \mid$$

$$\alpha r^2_{WP_k U_i} + (1-\alpha)\, r^2_{WP_k U_j} \leq \alpha D^2_{iP_k} + (1-\alpha)\, D^2_{jP_k} < r^2_{WP_k} + (\alpha^2 - \alpha)(d_{iw} + d_{jw})^2 \,]\!]$$

We note that if a variable $U \sim N(\,\mu,\, \sigma^2\,)$ is truncated above and below such that $A \leq U \leq B$. then

$$E[\, U \mid A \leq U \leq B\,] = \mu + \frac{Z\left[\frac{A-\mu}{\sigma}\right] - Z\left[\frac{B-\mu}{\sigma}\right]}{\Phi\left[\frac{B-\mu}{\sigma}\right] - \Phi\left[\frac{A-\mu}{\sigma}\right]}\, \sigma\,,$$

where $Z(u) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{u^2}{2}}$ and $\Phi(u)$ is the standard normal integral evaluated at u.

In the first expression. $U = D^2_{ij}$. $B = (\, d_{iw} + d_{jw}\,)^2$, $A = 0$. $\mu = E[\, D^2_{ij}\,]$ and $\sigma = \sqrt{\text{Var}[\, D^2_{ij}\,]}$.

Thus

$$E[\, D^2_{ij} \mid D^2_{ij} \leq (\, d_{iw} + d_{jw}\,)^2] = \mu + \frac{Z\left[\frac{-\mu}{\sigma}\right] - Z\left[\frac{(\, d_{iw} + d_{jw}\,)^2 - \mu}{\sigma}\right]}{\Phi\left[\frac{(\, d_{iw} + d_{jw}\,)^2 - \mu}{\sigma}\right] - \Phi\left[\frac{-\mu}{\sigma}\right]}\, \sigma\,.$$

In the second expression, $U = \alpha D^2_{iP_k} + (1-\alpha)\, D^2_{jP_k}$, $A = \alpha r^2_{WP_k U_i} + (1-\alpha)\, r^2_{WP_k U_j}$, $B = r^2_{WP_k}$ $+ (\alpha^2 - \alpha)(d_{iw} + d_{jw})^2$, $\mu = \alpha \mu_{iP_k} + (1-\alpha)\, \mu_{jP_k}$, and $\sigma^2 = \alpha^2 \sigma^2_{iP_k} + (1-\alpha)^2 \sigma^2_{jP_k}$. The conditional expectation is evaluated substituting U, A, B, μ, and σ in the formula given above.

Once β is estimated, then X^s_{DGZ} and Y^s_{DGZ} can be computed and the separation from the shifted DGZ to the unit locations i and j can be determined. In this case, the shifted DGZ which meets the preclusion area criteria is shifted too far away from target unit i or j to be able to engage them as an aggregate target. The targets may be able to be engaged as single targets.

d. $D^2_{DP_k} \geq r^2_{WP_k U_{ij}}$ for all preclusion areas k, but $D^2_{DP_k} < r^2_{WP_k}$ for some k. and

$(\, X^s_{DGZ} - X_{iL}\,)^2 + (\, Y^s_{DGZ} - Y_{iL}\,)^2 \leq d^2_{iw}$, and

$(\, X^s_{DGZ} - X_{jL}\,)^2 + (\, Y^s_{DGZ} - Y_{jL}\,)^2 \leq d^2_{jw}$.

If this outcome occurs, the aggregate target formed from units i and j can be engaged with weapon w. but the DGZ will need to be shifted away from the aggregate DGZ to coordinates (X^s_{DGZ}, Y^s_{DGZ}) which are at a distance r_{WP_k} from the center of mass of preclusion area k. The mixing parameter β is calculated as explained in the previous paragraph. In this case, the shifted DGZ is not shifted too far away from target units i and j to be able to engage them as an aggregate target.

Determining the Marginal Distribution of the Actual Ground Zero (AGZ)

Once the DGZ is determined, if a unit is engaged by a nuclear weapon, that weapon will be aimed at the DGZ. However, the round will generally not impact at the DGZ due to inherent errors in the delivery system accuracy. The weapon system accuracy is described by a Circular Error Probable (CEP) centered at the DGZ, where there is a 50% probability that the round will impact within the CEP. Implicit within the determination of a CEP is an assumption that the round impact point (Actual Ground Zero or AGZ) is distributed as a bivariate normal distribution around the DGZ with equal variances in the X and Y direction and zero correlation. Also implicit within the CEP is an assumption that the mean round impact point is the DGZ. If we assume that the DGZ coordinates X_{DGZ} and Y_{DGZ} have marginal normal distributions for any specified DGZ, the coordinates of the AGZ associated with that DGZ are also distributed as normal. Let CX_{DGZ} and CY_{DGZ} denote the delivery system error in the x and y direction, respectively, for the specified DGZ located at coordinates (X_{DGZ}, Y_{DGZ}). Then the coordinates (X_{AGZ}, Y_{AGZ}) of the AGZ are:

$$X_{AGZ} = (X_{DGZ} + CX_{DGZ}) \sim N(\mu_{X_{DGZ}} + \mu_{CX_{DGZ}}, \sigma^2_{X_{DGZ}} + \sigma^2_{CX_{DGZ}}),$$
$$Y_{AGZ} = (Y_{DGZ} + CY_{DGZ}) \sim N(\mu_{Y_{DGZ}} + \mu_{CY_{DGZ}}, \sigma^2_{Y_{DGZ}} + \sigma^2_{CY_{DGZ}}).$$

NOTE: Because of the CEP assumptions stated above, $\mu_{CX_{DGZ}} = \mu_{CY_{DGZ}} = 0$ and $\sigma^2_{CX_{DGZ}} = \sigma^2_{CY_{DGZ}}$.

Alternative Calculations

An alternative to numerical calculations or approximations for the probabilities of preclusion, aggregation, etc. is to solve them through a statistical Monte Carlo approach. It is relatively easy to generate a set of random variables that are jointly distributed as multinormal (Law and Kelton [1982], see also Appendix C). Thus a simple statistical simulation can be set up, given the unit template and variance/covariance matrices, that generates a complete set of unit locations per replication. For each replication, the separation distances between the units and between the units and preclusion areas can be calculated to estimate the probability that units are close enough together to aggregate and are close enough to preclusion areas to cause a DGZ shift. In other words, we can directly estimate quantities such as the following.

$\overline{p}_{prec}(i | w, a_i, a\overline{g}g_i) \doteq P[\bigcap_k D^2_{iP_k} > r^2_{WP_kU_i}]$, the probability that the target unit will *not* be precluded from engagement by weapon w.

$\overline{p}_{aggr}(i | w, a_i) \equiv P[\{ \bigcup_j \{ (D^2_{ij} \geq d^2_{ijw}) \cap (j \text{ available}) \cap (\bigcap_k D^2_{iP_k} > r^2_{WP_kU_i}) \} \}^c | w, a_i]$, the probability that the unit i will *not* be aggregated using weapon w.

We can also estimate the probabilities of DGZ shift, etc. as outlined in the section in Chapter 3 on "*Categorizing Possible Outcomes*".

It is also possible to quickly determine if a unit can be defeated with a weapon w for each unit in each replication (Chapter 3). This would be a function of aggregation and preclusion; obviously if weapon w was precluded from engaging unit i in a replication, it cannot be used to defeat i in that replication. The result of this calculation, averaged across all replications, is an estimate of

$$p_{defeat}(\ i \mid w) = P[\ \text{Unit } i \text{ defeated} \mid \text{weapon } w \text{ used }]\ .$$

The probability that unit i can be defeated is

$$p_{defeat}(\ i\ ) = \sum_w P[\ \text{Unit } i \text{ defeated} \mid \text{weapon } w \text{ used }]\ P[\ \text{weapon } w \text{ used }].$$

At first glance, this approach may seem the same as running a standard nuclear exchange simulation model. The approach is similar, but offers many advantages over the simulation model. First, the statistical simulation used here is very simple. No events, timing, acquisition, etc. needs to be represented; the calculations consist merely of generating a set of correlated pseudorandom numbers, calculating a set of distances, and comparing these distances to values such as r_{WP_k} and $d_{i,w}$. Second, we can easily break the calculations into parts, to determine intermediate results such as $\bar{p}_{prec}(\ i \mid w, a_i, a\bar{g}g_i\ )$, which enable us to gain a greater understanding of what is happening in the model. Third, we can use these probabilities directly in simple probability calculations to determine such values as $p_{defeat}(\ i\ )$. The approach described in this paper allows us to see the sensitivity of the results to various factors, such as acquisition, movement, location, weapon, preclusion areas, etc. very transparently. Finally, once the calculations of location-based values such as $\bar{p}_{prec}(\ i \mid w, a_i, a\bar{g}g_i\ )$, $\bar{p}_{aggr}(\ i \mid w, a_i\ )$, and $p_{defeat}(\ i \mid w)$ have been made, it is possible to perform all of the analysis currently made using the detailed simulation models without having to recalculate these probabilities. The reason this is true is that current simulation models, using manually generated arrays or static templates, implicitly assume that the unit locations do not change over the various excursions that are run using the model. We can make a much weaker assumption that the *distribution parameters* describing the location of the units do not change, and proceed to run all of the same types of analyses as can be run on the detailed simulation model.

An algorithm for estimating the probability parameters using Monte Carlo techniques is provided in Appendix E.

CHAPTER 3
MODELING THE IMPACT OF THE PROBABILISTIC ARRAYS
ON THE NUCLEAR EXCHANGE

Section I. CHAPTER SUMMARY

Summary of Contents

In this chapter, we will determine the probabilities associated with each of the engagement possibilities discussed in Chapter 2. Then, using the distributions of the actual unit locations and the weapon AGZs, we will determine the probability that a given unit can be defeated using each type weapon. A scheme of allocating weapons to targets is discussed resulting in a probability for each unit that a round of type w will be allocated against the unit. Finally, we will average across all of the available weapon systems to yield the probability that each unit can be defeated. All combinations of defeat / failure to defeat the individual units thus forms the set of all possible outcomes of a nuclear exchange.

Summary of the Logic

In order to understand the probability structure, it is important to keep in mind two elementary facts. If we denote events by A and B (an example of an event is the unit i being located at a distance greater than the preclusion distance away from all preclusion areas, for a given weapon w):

(1) The Multiplication Rule: $P[\ A,\ B\ ] = P[\ A\ |\ B\ ]P[\ B\ ]$, where the notation "$P[\ A\ |\ B\ ]$" denotes the probability of event A, given that event B has happened (or is true).

(2) The Addition Rule: $P[\ A\ ] = \sum_{all\ i} P[\ A\ |\ B_i\ ]P[\ B_i\ ] = \sum_{all\ i} P[\ A,\ B_i\ ]$, where the events B_i are exhaustive and mutually exclusive (that is, only on B_i can be true and at least one B_i is true).

An example of the first rule is found in the probability that unit i can be engaged as a single target with a DGZ shift. $P[\ unit\ i$ engaged as a single target with a DGZ shift $|$ weapon w is used $]$ = $P[\ unit\ i$ engaged as a single target $|$ weapon w is used and the DGZ is shifted $]\ \cdot\ P[\ DGZ$ is shifted $|$ weapon w is used $]$. An example of the second rule is found in the fact that if a unit is

engaged as a single target, it must be engaged with a DGZ shift or without a DGZ shift - thus the events of DGZ shift / no DGZ shift are exhaustive and mutually exclusive. The probability that unit i can be engaged as a single target using weapon w is : P[unit i engaged as a single target | weapon w is used] = P[unit i engaged as a single target with a DGZ shift | weapon w is used] + P[unit i engaged as a single target without a DGZ shift | weapon w is used].

A particular notation is used to keep track of which events are of interest and which events are given.

(1) Each unit i has a probability $p_{avail}(i)$ that it is available. If the availability is given, we use the notation a_i to denote this. The notation a_{ij} is used to denote the fact that both units i and j are available.

(2) Each unit i may or may not be aggregated. If unit i is aggregated, the notation agg_i is used. If unit i is aggregated with unit j, the notation agg_{ij} is used.

(3) The DGZ for a particular weapon type may or may not be shifted. If the DGZ is shifted, we use the notation s. The notation $\bar{s}$ is used when the DGZ is not shifted.

(4) Initially, we condition all of the probabilities on a particular weapon type w, denoted by w.

(5) We denote the opposite of an event (more technically, the complement) by a bar over the notation. Thus for example $\bar{s}$ denotes "not s" or not shifted, and $\overline{agg}_i$ denotes the event that unit i is not aggregated.

(6) We denote the opposite of an probability (more technically, 1 minus the probability or the probability of the complement) by a bar over the "p" in the notation. For example if $p_{prec}(i|w, a_i, \overline{agg}_i)$ denotes the probability that unit i is precluded, given that weapon w is used, unit i is available and not aggregated, then $\bar{p}_{prec}(i|w, a_i, \overline{agg}_i)$ denotes the probability that unit i is *not* precluded, given that same information.

Recall each of the engagement probabilities from the previous chapter. If a target unit i is available, then for a particular weapon type w unit i may be:

(1) Engaged as a single target with no DGZ shift. We denote the probability that this occurs as $p_{engage}(i, \overline{s} \mid w)$.

(2) Engaged as a single target with a DGZ shift. We denote the probability that this occurs as $p_{engage}(i, s \mid w)$.

(3) Engaged as part of an aggregate target (with unit j) with no DGZ shift. We denote the probability that this occurs as $p_{engage}(ij, \overline{s} \mid w)$.

(4) Engaged as part of an aggregate target (with unit j) with a DGZ shift. We denote the probability that this occurs as $p_{engage}(ij, s \mid w)$.

(5) Not engaged. Since this will occur if unit i is precluded from fire by a preclusion area, we denote the probability that this occurs as $p_{prec}(i \mid w, a_i, a\overline{g}g_i)$. Unit i cannot be considered for aggregation if it is precluded, so $p_{prec}(i \mid w, a_i, agg_i) = \overline{p}_{prec}(i \mid w, a_i, agg_i) = 0$.

In the chapter, we determine the intermediate probabilities (for example, the probability of a single target DGZ shift $p_{shift}(i \mid w, a_i, a\overline{g}g_i)$) step by step as shown in Figures 11, 12, and 15 and combine them to get the engagement probabilities for all weapons w. The probability that a round of type w is allocated against unit i, denoted as $p_{round}(w \mid a_i)$, is used to determine the probability that unit i is engaged (as one of the four engagement possibilities) *and* weapon w is used to engage it.

Figure 15 shows how we can complete our determination of the defeat probabilities. For each of the engagement possibilities, we compute the AGZ to actual unit location distribution and determine the probability of defeat conditioned on that engagement possibility. Multiplied by the engagement probability, this yields the probabilities $p_{defeat}(i, w, a_i, \overline{s}, a\overline{g}g_i)$, $p_{defeat}(i, w, a_i, s, a\overline{g}g_i)$, $p_{defeat}(i, w, a_{ij}, s, agg_{ij})$, and $p_{defeat}(i, w, a_{ij}, \overline{s}, agg_{ij})$ that unit i can be defeated. At the bottom of Figure 15, we also compute the probability that unit i is defeated as a bonus target by a round aimed at unit j, for all $j \neq i$. We can add these probabilities using the addition rule to get $p_{defeat}(i, w)$, the probability that unit i is defeated and weapon w is allocated to engage it. If we apply the addition rule once more, we can sum over all w to get the net probability that unit i is defeated by a nuclear weapon, $p_{defeat}(i)$.

Summary of the Results

For a potential unit i, the intermediate probabilities of interest are:

(1) The probability that it is available, $p_{avail}(i)$

(2) The probability that it will not be engaged as an aggregate target, $\overline{p}_{aggr}(i|w, a_i)$.

(3) The probability that it can be engaged as an aggregate target with a unit j, $p_{aggr}(ij|w, a_{ij})$.
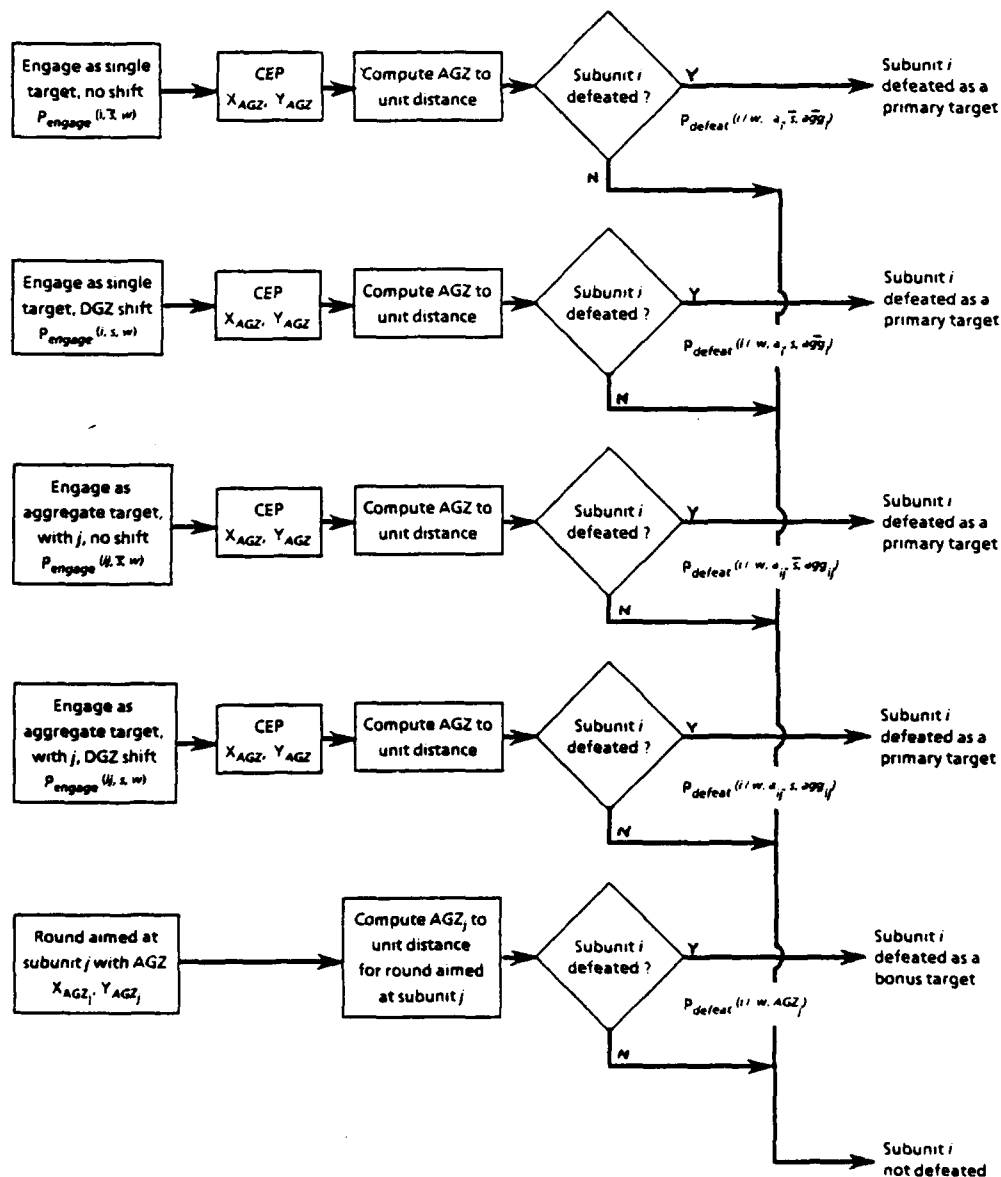
Figure 15. Logic for Determining the Defeat Probabilities for Unit i

(4) The probability that the DGZ will not be shifted if it is engaged as a single target.

$P_{no\ shift}(i \mid w, a_i, a\overline{g}g_i)$.

(5) The probability that the DGZ will be shifted if it is engaged as a single target.

$P_{shift}(i \mid w, a_i, a\overline{g}g_i)$.

(6) The probability that the DGZ will not be shifted if it is engaged as an aggregate target,

$P_{no\ shift}(ij \mid w, a_{ij}, agg_{ij})$.

(7) The probability that the DGZ will be shifted if it is engaged as an aggregate target.

$P_{shift}(ij \mid w, a_{ij}, agg_{ij})$.

If a target unit i is available, then the probabilities that it may be engaged with a particular weapon type w follow:

(1) Engaged as a single target with no DGZ shift.

$$P_{engage}(i, \overline{s} \mid w) = P_{avail}(i) \cdot P_{no\ shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{P}_{aggr}(i \mid w, a_i)$$

(2) Engaged as a single target with a DGZ shift.

$$P_{engage}(i, s \mid w) = P_{avail}(i) \cdot P_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{P}_{aggr}(i \mid w, a_i)$$

(3) Engaged as part of an aggregate target with no DGZ shift.

$$P_{engage}(ij, \overline{s} \mid w) = P_{avail}(i) \cdot P_{avail}(j) \cdot P_{no\ shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot P_{aggr}(ij \mid w, a_{ij})$$

(4) Engaged as part of an aggregate target with a DGZ shift.

$$P_{engage}(ij, s \mid w) = P_{avail}(i) \cdot P_{avail}(j) \cdot P_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot P_{aggr}(ij \mid w, a_{ij})$$

(5) Precluded from engagement.

$$P_{prec}(i \mid w, a_i, a\overline{g}g_i).$$

Given the four engagement types (plus a bonus target possibility) for a particular weapon type w, the following conditional defeat probabilities can be computed:

(1) Engaged as a single target with no DGZ shift.

$$P_{defeat}(i \mid w, a_i, \overline{s}, a\overline{g}g_i) = P[(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2 \leq d_{iw}^2]$$

(2) Engaged as a single target with a DGZ shift.

$$P_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) = P[(X_{AGZ}^s - X_i)^2 + (Y_{AGZ}^s - Y_i)^2 \leq d_{iw}^2].$$

(3) Engaged as part of an aggregate target with no DGZ shift.

$$P_{defeat}(i \mid w, a_{ij}, \overline{s}, agg_{ij}) = P[(X_{AGZ}-X_i)^2+(Y_{AGZ}-Y_i)^2 \leq d_{iw}^2].$$

(4) Engaged as part of an aggregate target with a DGZ shift.

$$P_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) = P[(X_{AGZ}^s-X_i)^2+(Y_{AGZ}^s-Y_i)^2 \leq d_{iw}^2].$$

(5) Defeated as a bonus target for round aimed at unit j $\forall$ $j \neq i$.

$$P_{defeat}(i \mid j \text{ engaged}, w, a_j) = \sum_{all \ j \neq i} P[(X_{AGZ_j}-X_i)^2+(Y_{AGZ_j}-Y_i)^2 \leq d_{iw}^2].$$

The unconditional defeat probabilities can be computed directly from the above.

The remainder of this chapter simply presents the mathematics behind the probability calculations.

Section II. DERIVATIONS AND RESULTS

Introduction

Up until now, we have computed the probability distributions for target locations, perceived target locations, and the distribution of the DGZ given no shift from perceived target center or given a shift due to aggregation and/or preclusion. If we simply wish to estimate the locations of the DGZ's (assuming the target units were acquired and weapons are available for fire), our task would be done. However, if we wish to extend the analysis to estimate the impact of the probability distributions for the nuclear force arrays on the theater-level nuclear exchange, we must determine the sample space of all possible outcomes of the exchange and determine the probability associated with each possible outcome. In order to get results, it will be necessary to adopt some simplistic heuristics for the joint probabilities of aggregation and preclusion. Using these heuristics will enable us to at least roughly estimate the impact of the unit locations on the nuclear exchange.

Point Targets

The methodology developed in this paper for area targets may also be used against point targets. Point target defeat criteria are normally stated as an X% assurance that a point target will receive a specified degree of damage, rather than an X% assurance that at least Y% of an area target will receive a specified degree of damage. An equivalent way of expressing this point target criteria is

to require an X% probability that the radius associated with the desired degree of damage will overlap the point target. Another equivalent formulation is the requirement that the distance between the DGZ and the point target is less than the radius associated with the desired degree of damage with a probability of X%.

The last formulation is similar to the defeat criteria for area targets. If we define a point target as a circular area target with an arbitrarily small radius, and a defeat criteria of 100% coverage, the probability that this area target is defeated is the same as the probability of defeating the equivalent point target. Thus, we model all point targets as area targets with very small radii and apply the methodology previously described.

Evaluating Joint Probabilities

Determining the probability of aggregation, and the probability that any pair of targets can be aggregated, involves a calculation of the joint distribution of all of the separation distances between units. This evaluation of joint probabilities also appears when we calculate the probability that one or more preclusion areas may cause a DGZ shift or eliminate a potential target from engagement. Our approach to this general problem is illustrated using the example of aggregation. We will use the following notation:

Let $D_{ij}^2 = [X_{iL} - X_{jL}]^2 + [Y_{iL} - Y_{jL}]^2$, the squared distance between the perceived locations of units i and j.

Let d_{iw} = the maximum distance that a weapon w can be displaced from the center of mass of a targeted unit i and still achieve the commander's defeat criteria.

Let d_{ijw} = the maximum distance between the center of mass of targets i and j that will permit weapon w to be used to cover both targets and achieve at least the commander's defeat criteria. Clearly,

$$d_{ijw} = d_{iw} + d_{jw}.$$

Let $G_{ijw} = 1$ if $D_{ij}^2 \leq d_{ijw}^2$ (thus targets i and j can be aggregated using w),

         $= 0$ otherwise.

Let $\{ A_j \}_{j=1}^m$ denote the set $\{ A_1, A_2, ..., A_m \}$.

Let the subscript w (the weapon type) go from 1 to nw.

We also note the following inequalities: for any events A and B,

$P(A \cap B) \leq \min[P(A), P(B)]$

$\max[P(A), P(B)] \leq P(A \cup B) \leq P(A) + P(B)$

Thus

$$P(\text{ no aggregation }) = P[\bigcap_{w=1}^{nw} \bigcap_{i,j=1}^{m} G_{ijw} = 0] \le \min\Big\{ \{ P[\, G_{ijw} = 0 \,] \}_{i,j=1}^{m} \Big\}_{w=1}^{nw} , \text{ and}$$

$$P(\text{ aggregation }) = P[\bigcup_{w=1}^{nw} \bigcup_{i,j=1}^{m} G_{ijw} = 1] \le \sum_{w=1}^{nw} \sum_{i,j=1}^{m} P[\, G_{ijw} = 1 \,] ; \text{ thus}$$

$$\max\Big\{ \{ P[\, G_{ijw} = 1 \,] \}_{i,j=1}^{m} \Big\}_{w=1}^{nw} \le P(\text{ aggregation }) \le \sum_{w=1}^{nw} \sum_{i,j=1}^{m} P[\, G_{ijw} = 1 \,].$$

Note that unless m, nw, and the probabilities $P[\, G_{ijw} = 1 \,]$ are small, the upper bound will be greater than 1 and thus trivial. We can also compute the following:

$P[\text{ only 1 aggregation (between units } i \text{ and } j) \text{ using weapon } w\,]$

$$\le \min \left[P[\, G_{ijw} = 1 \,], \{ P[\, G_{klw} = 0 \,] \}_{\substack{k,l=1 \\ (k,l)\neq(i,j)}}^{m} \right] \le P[\, G_{ijw} = 1 \,].$$

Notice also that

$$P[\, 2 \text{ aggregations between units } (\, i, j\,) \text{ and } (\, k, l\,)\,] \le \min \left[P[\, G_{ijw} = 1 \,], P[\, G_{klw} = 1 \,] \right],$$

thus

$$P[\, 2 \text{ aggregations between units } (\, i, j\,) \text{ and } (\, k, l\,)\,] \le P[\, G_{ijw} = 1 \,], \text{ and}$$

$$P[\, 2 \text{ aggregations between units } (\, i, j\,) \text{ and } (\, k, l\,)\,] \le P[\, G_{klw} = 1 \,] ,$$

for any units i, j, k and l and weapon w.

As a result, we can provide an upper bound on the probability of aggregation occurring between any units i and j using weapon w by simply looking at $P[\, G_{ijw} = 1 \,]$, which can be evaluated from the marginal distribution of D_{ij}^2.

Let $(\, i,j,w\,)_{(1)} = \Big\{ (\, i,j,w\,) \mid P[\, G_{(i,j,w)_{(1)}} = 1 \,] = \max_{i,j} P[\, G_{ijw} = 1 \,] \Big\}$. Let $(\, i,j,w\,)_{(2)}$ denote the indices of the second largest probability of an aggregation, etc. We can order the upper bounds on the probability that unit pair $(\, i,j\,)$ forms an aggregate pair for all i,j,w as:

$$P[\, G_{(i,j,w)_{(1)}} = 1 \,] \ge P[\, G_{(i,j,w)_{(2)}} = 1 \,] \ge \cdots \ge P[\, G_{(i,j,w)_{(m)(m-1)}} = 1 \,].$$

We will use this approach to estimate which target unit pairs $(\, i,j\,)$ to consider as aggregate targets.

If we approximate $P(\text{ no aggregation unit } i \mid w\,)$ using its upper bound $\min\{\, P[G_{ijw} = 0\,]\,\}_{\substack{j=1 \\ j\neq i}}^{m}$, we can avoid the problem that can arise from independence assumptions. For example, suppose that there are a very large number of units ($m = 100$, say), with all of the units having expected (mean)

locations far away from the expected location of unit i. We would expect the probability that $D_{ij}^2 > d_{ijw}$ for each unit j with a mean location far away from the mean location of unit i to be very large, say 0.99. However, P(no aggregation unit $i \mid w$) under independence is $(0.99)^{100} = 0.37$. In actual arrays, the units locations are correlated and we would expect aggregation to occur very infrequently, certainly less than (100-37) = 63% of the time! With independence assumptions, adding units with locations very far away from the unit of interest can make the probability of no aggregation very small, which contradicts experience. Using the upper bound of $\min\{ P[G_{ijw} = 0] \}_{j=1}^{m}$, $j \neq i$, on the other hand, will ignore the units far away from the unit of interest (only the unit that is the most likely to be close with the minimum $P[G_{ijw} = 0]$ is considered).

Some random distance pairs will be more dependent than others. The distances D_{ij} and D_{ik} are strongly dependent as they both involve the location of unit i. The distances D_{ij} and D_{kl}, $(i,j) \neq (k,l)$, on the other hand, are less dependent as they are related only through the covariances between i, j, k, and l. If the distances are *associated* (roughly speaking, having positive covariances), then a lower bound for the joint probabilities is the product of the two (that is, the independent case). We distinguish between the two cases in our heuristic rules for evaluating joint probabilities. We also assume that there is a difference between evaluations of distances computed from different random points for aggregation and distances computed from different fixed points for preclusion.

Our heuristic rules for evaluating joint probabilities are as follows.

1. For multiple comparisons of distances from the same point, use the upper bound of the smallest distance. Example: $P[\bigcap_k D_{iP_k}^2 > r_{WP_kU_i}^2] \doteq \min\{ P[D_{iP_k}^2 > r_{WP_kU_i}^2] \}_{k=1}^{np}$.

2. For multiple comparisons of distances from two different points, use the approximate lower bound of the product of the distances. Example:

$$P[\{ D_{ij}^2 < (d_{iw} + d_{jw})^2 \} \cap \{ \bigcap_k D_{DP_k}^2 > r_{WP_kU_{ij}}^2 \}]$$
$$\doteq P[D_{ij}^2 < (d_{iw} + d_{jw})^2] \cdot P[\bigcap_k D_{DP_k}^2 > r_{WP_kU_{ij}}^2]$$
$$\doteq P[D_{ij}^2 < (d_{iw} + d_{jw})^2] \cdot \min\{ P[D_{DP_k}^2 > r_{WP_kU_{ij}}^2] \}_{k=1}^{np}.$$

3. For comparisons between distances computed from different random points for aggregation and different fixed points for preclusion, use the approximate lower bound of the product of the distances. Example: D_{ij} is the distance between the random points of the locations of

units i and j (for aggregation), while D_{iP_k} is the distance between the random point of the location of unit i and the fixed points of the preclusion areas (for preclusion). Example:

$$P[\left\{ D_{ij}^2 < (d_{iw} + d_{jw})^2 \right\} \cap \left\{ \bigcap_k D_{iP_k}^2 > r_{WP_kU_i}^2 \right\}]$$

$$\doteq P[D_{ij}^2 < (d_{iw} + d_{jw})^2] \cdot P[\bigcap_k D_{iP_k}^2 > r_{WP_kU_i}^2]$$

$$\doteq P[D_{ij}^2 < (d_{iw} + d_{jw})^2] \cdot \min_k \left\{ P[D_{iP_k}^2 > r_{WP_kU_i}^2] \right\}_{k=1}^{np}.$$

Although these heuristics seem reasonable as approximations, the error of approximation is not known. We are conducting some simulation experiments to verify these rules and estimate the error of approximation.

Determining the Probabilities Related to Location

For every unit i, compute the following:

1. Calculate $p_{avail}(i)$, the probability that a unit i is acquired and retained as a target until detonation, as explained in Chapter 2.

2. For every preclusion area k, compute $P[D_{iP_k}^2 > r_{WP_kU_i}^2]$ and $P[D_{iP_k}^2 > r_{WP_k}^2]$.

   a. Let $\bar{p}_{prec}(i \mid w, a_i, \overline{agg}_i)$, the probability that the unit will *not* be precluded from engagement, given that it is available, can be engaged using weapon w, and it is not suitable for engagement as an aggregate target; i.e., $D_{ij}^2 > (d_{iw} + d_{jw})^2 = d_{ijw}^2$ or j was not available $\forall\, j \neq i$. However, when we approximate joint probabilities, we assume that

   $P[\left\{\bigcap_k D_{iP_k}^2 > r_{WP_k}^2\right\} \cap \left\{D_{ij}^2 > (d_{iw} + d_{jw})^2\right\}] = P[\bigcap_k D_{iP_k}^2 > r_{WP_k}^2] \cdot P[D_{ij}^2 > (d_{iw} + d_{jw})^2]$, thus

   $P[\bigcap_k D_{iP_k}^2 > r_{WP_k}^2 \mid D_{ij}^2 > (d_{iw} + d_{jw})^2] = P[\bigcap_k D_{iP_k}^2 > r_{WP_k}^2]$ and we ignore the conditioning

   when evaluating $P[\bigcap_k D_{iP_k}^2 > r_{WP_k}^2]$. Thus

   $\bar{p}_{prec}(i \mid w, a_i, \overline{agg}_i) \equiv P[$ unit i will *not* be precluded $\mid$ available, weapon w, $D_{ij}^2 > d_{ijw}^2]$.

   $\bar{p}_{prec}(i \mid w, a_i, \overline{agg}_i) \doteq P[\bigcap_k D_{iP_k}^2 > r_{WP_kU_i}^2]$, bounded from above as:

   $\bar{p}_{prec}(i \mid w, a_i, \overline{agg}_i) \doteq \min_k \left\{ P[D_{iP_k}^2 > r_{WP_kU_i}^2] \right\}_{k=1}^{np}.$

b. Let $p_{no\ shift}(i|w, a_i, a\overline{g}g_i)$ be the probability that there is no shift in DGZ (due to preclusion) for target i, given that weapon w is used, unit i is available, and there are no suitable aggregate targets. By definition, if there is no DGZ shift due to preclusion, the unit i is not precluded. As in $\overline{p}_{prec}(i|w, a_i, a\overline{g}g_i)$, we condition on $D_{ij}^2 > (d_{iw} + d_{jw})^2$ but ignore the conditioning when evaluating the joint probabilities. Thus $p_{no\ shift}(i|w, a_i, a\overline{g}g_i) \equiv P[$ no shift in DGZ for target i | weapon w, i available, no aggregation $] = P[\underset{k}{\cap} D_{iP_k}^2 > r_{WP_k}^2]$

$$p_{no\ shift}(i|w, a_i, a\overline{g}g_i) \doteq \min_k \left\{ P[D_{iP_k}^2 > r_{WP_k}^2] \right\}_{k=1}^{np}.$$

c. Let $p_{shift}(i|w, a_i, a\overline{g}g_i)$ be the probability that target unit i can be engaged as a single target with a shift in DGZ due to preclusion, given that weapon w is used and unit i is available. Again, we condition on no suitable aggregate targets, $D_{ij}^2 > (d_{iw} + d_{jw})^2$ but ignore the conditioning when evaluating the joint probabilities.

$$p_{shift}(i|w, a_i, a\overline{g}g_i) = P\left(\left\{ \underset{k}{\cap} D_{iP_k}^2 > r_{WP_kU_i}^2 \right\} \cap \left\{ \underset{k}{\cap} D_{iP_k}^2 > r_{WP_k}^2 \right\}^C \right)$$

$$= P[\underset{k}{\cap} D_{iP_k}^2 > r_{WP_kU_i}^2] - P[\underset{k}{\cap} D_{iP_k}^2 > r_{WP_k}^2]$$

$$p_{shift}(i|w, a_i, a\overline{g}g_i) \doteq \min_k \left\{ P[D_{iP_k}^2 > r_{WP_kU_i}^2] \right\}_{k=1}^{np} - \min_k \left\{ P[D_{iP_k}^2 > r_{WP_k}^2] \right\}_{k=1}^{np}$$

Note that

$$p_{shift}(i|w, a_i, a\overline{g}g_i) + p_{no\ shift}(i|w, a_i, a\overline{g}g_i)$$
$$= P[\underset{k}{\cap} D_{iP_k}^2 > r_{WP_kU_i}^2] = \overline{p}_{prec}(i|w, a_i, a\overline{g}g_i)$$

3. For every unit j, $j \neq i$, compute $P[D_{ij}^2 \leq (d_{iw} + d_{jw})^2 = d_{ijw}^2]$.

a. Let $G_{ijw} = 1$ if $D_{ij}^2 \leq d_{ijw}^2$, given weapon w and both units i, j available.
         $= 0$ otherwise.

Then

$$\overline{p}_{aggr}(w|\text{available}) \equiv P(\text{no aggregation with weapon } w | \text{all units available})$$
$$= P[\underset{\substack{i,j=1 \\ j \neq i}}{\overset{m}{\cap}} G_{ijw} = 0]$$

$$\overline{p}_{aggr}(w|\text{available}) \doteq \min_{i,j} \left\{ P[G_{ijw} = 0] \right\}_{i,j=1}^{m}$$

b. Let $p_{aggr}(ij|w, a_{ij})$ be the probability that units i and j can be aggregated, given that weapon w is used and units i and j are available.

$$p_{aggr}(ij|w, a_{ij}) \equiv P[D_{ij}^2 \leq (d_{iw} + d_{jw})^2 | w; i, j \text{ available}] = P[G_{ijw} = 1]$$

c. Let $\overline{p}_{aggr}(ij|w, a_{ij})$ be the probability that units i and j *cannot* be aggregated, given that weapon w is used and units i and j are available.

$$\overline{p}_{aggr}(ij|w, a_{ij}) \equiv P[\,D_{ij}^2 > (d_{iw} + d_{jw})^2\,|\,w;\,i \text{ and } j \text{ available}\,]$$
$$= P[\,G_{ijw} = 0\,] = 1 - p_{aggr}(ij|w, a_{ij})$$

d. Let $\overline{p}_{aggr}(i|w, a_i)$ be the probability that *no* unit can be aggregated with unit i, given that weapon w is used and unit i is available. This can occur when $D_{ij}^2 > d_{ijw}^2$ or when j is not available for fire planning.

$$\overline{p}_{aggr}(i|w, a_i) \equiv P(\text{no aggregation involving unit } i \mid \text{weapon } w; i \text{ available})$$
$$= P[\{\underset{j}{\cup}\{(D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge r_{WP_k U_j})\}\}^c \mid w, a_i]$$
$$= P[\underset{j}{\cap}\{(D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge r_{WP_k U_j})\}^c \mid w, a_i]$$
$$\doteq \underset{j}{\min}\left\{P[\{(D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge r_{WP_k U_j})\}^c \mid w, a_i]\right\}_{\substack{j=1 \\ j \ne i}}^{m}$$

$$\overline{p}_{aggr}(i|w, a_i) \doteq \underset{j}{\min}\left\{1 - P[(D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge r_{WP_k U_j}) \mid w, a_i]\right\}_{\substack{j=1 \\ j \ne i}}^{m}$$

$$\doteq 1 - \underset{j}{\max}\left\{P[D_{ij} \le d_{ijw} \mid w, a_{ij}] \cdot P[j \text{ available}] \cdot P[\underset{k}{\cap} D_{jP_k} \ge r_{WP_k U_j} \mid w, a_{ij}]\right\}_{\substack{j=1 \\ j \ne i}}^{m}$$

$$\doteq 1 - \underset{j}{\max}\left\{p_{aggr}(ij|w, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j|w, a_j, \overline{agg}_j)\right\}_{\substack{j=1 \\ j \ne i}}^{m}$$

4. Find j such that j solves $\underset{j}{\max}\left\{p_{aggr}(ij|w, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j|w, a_j, \overline{agg}_j)\right\}_{\substack{j=1 \\ j \ne i}}^{m}$. This will be the unit j that may serve as an aggregate target (along with unit i).

a. For the target pair (i,j), compute the proportion α such that the aggregate target DGZ has coordinates $X_{DGZ} = \alpha X_{iL} + (1 - \alpha) X_{jL}$ and $Y_{DGZ} = \alpha Y_{iL} + (1 - \alpha) Y_{jL}$. The formulas for computing α were given in Chapter 2.

b. For every preclusion area k, compute $P[\,D_{DP_k}^2 > r_{WP_k U_i}^2\,]$ and $P[\,D_{DP_k}^2 > r_{WP_k}^2\,]$, where $D_{DP_k}^2 =$ the distance from the unshifted aggregate target (X_{DGZ}, Y_{DGZ}) to preclusion area k; that is, $D_{DP_k}^2 = (X_{DGZ} - X_{P_k})^2 + (Y_{DGZ} - Y_{P_k})^2$.

c. Let $\overline{p}_{prec}(ij\,|\,w,a_{ij},agg_{ij})$ equal the probability that *both* target units i and j are *not* precluded from engagement, given that both are available, both can be aggregated with each other, and they can be jointly engaged using weapon w. In this case, we condition on $D_{ij}^2 \leq d_{ij\,w}^2$. Thus

$$\overline{p}_{prec}(ij\,|\,w,a_{ij},agg_{ij}) \equiv P[\,\bigcap_k \{\, D_{DP_k}^2 > r_{WP_kU_i}^2 \cap D_{DP_k}^2 > r_{WP_kU_j}^2 \,\}\,|\,D_{ij}^2 \leq d_{ij\,w}^2\,].$$

We approximate as

$$\overline{p}_{prec}(ij\,|\,w,a_{ij},agg_{ij}) \doteq \min_k \left\{\, P[\, D_{DP_k}^2 > r_{WP_kU_{ij}}^2 \,|\, D_{ij}^2 \leq d_{ij\,w}^2]\,\right\}_{k=1}^{np}.$$

To evaluate $P[\, D_{DP_k}^2 > r_{WP_kU_{ij}}^2 \,|\, D_{ij}^2 \leq d_{ij\,w}^2]$ for any unit pairs i,j, we recall that

$$D_{DP_k}^2 = (\alpha^2 - \alpha)\, D_{ij}^2 + (1 - \alpha)\, D_{jP_k}^2 + \alpha\, D_{iP_k}^2.$$

We know that $E[D_{DP_k}^2] = (\alpha^2 - \alpha)\, E[D_{ij}^2] + (1 - \alpha)\, E[D_{jP_k}^2] + \alpha\, E[D_{iP_k}^2]$, and it can be shown that $\mathrm{Var}[D_{DP_k}^2] = (\alpha^2-\alpha)^2\, \mathrm{Var}[D_{ij}^2] + (1-\alpha)^2\, \mathrm{Var}[\overline{D}_{jP_k}^2] + \alpha^2\, \mathrm{Var}[D_{iP_k}^2] + 2(\alpha^2-\alpha)(1-\alpha)\,\mathrm{Cov}[D_{ij}^2, D_{jP_k}^2] + 2(\alpha^2-\alpha)\alpha\,\mathrm{Cov}[D_{ij}^2, D_{iP_k}^2] + 2\alpha(1-\alpha)\,\mathrm{Cov}[D_{iP_k}^2, D_{jP_k}^2]$. If we make a simplification for the purpose of evaluation by assuming 1) that $D_{DP_k}^2$ is distributed normally (i.e., use the normal approximation to the distribution of a quadratic form) and 2) that D_{ij}^2, $D_{jP_k}^2$, and $D_{iP_k}^2$ are independent, then $D_{DP_k}^2$ given $D_{ij}^2 \leq d_{ij\,w}^2$ is approximately normal with:

Mean: $(\alpha^2-\alpha)\,E[D_{ij}^2\,|\,D_{ij}^2 \leq d_{ij\,w}^2] + (1-\alpha)\,E[D_{jP_k}^2] + \alpha\,E[D_{iP_k}^2]$ and

Variance: $(\alpha^2-\alpha)^2\,\mathrm{Var}[D_{ij}^2\,|\,D_{ij}^2 \leq d_{ij\,w}^2] + (1-\alpha)^2\,\mathrm{Var}[D_{jP_k}^2] + \alpha^2\,\mathrm{Var}[D_{iP_k}^2]$.

To evaluate, recall that if a variable $U \sim N(\mu, \sigma^2)$ is truncated above and below such that $A \leq U \leq B$, then

$$E[\,U\,|\,A \leq U \leq B\,] = \mu + \frac{Z\left[\frac{A-\mu}{\sigma}\right] - Z\left[\frac{B-\mu}{\sigma}\right]}{\Phi\left[\frac{B-\mu}{\sigma}\right] - \Phi\left[\frac{A-\mu}{\sigma}\right]}\,\sigma \quad \text{and}$$

$$\mathrm{Var}[\,U\,|\,A \leq U \leq B\,] = \left\{ 1 + \frac{\left[\frac{A-\mu}{\sigma}\right]Z\left[\frac{A-\mu}{\sigma}\right] - \left[\frac{B-\mu}{\sigma}\right]Z\left[\frac{B-\mu}{\sigma}\right]}{\Phi\left[\frac{B-\mu}{\sigma}\right] - \Phi\left[\frac{A-\mu}{\sigma}\right]} \right.$$

$$\left. - \left[\frac{Z\left[\frac{A-\mu}{\sigma}\right] - Z\left[\frac{B-\mu}{\sigma}\right]}{\Phi\left[\frac{B-\mu}{\sigma}\right] - \Phi\left[\frac{A-\mu}{\sigma}\right]}\right]^2 \right\}\sigma^2$$

where $Z(u) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{u^2}{2}}$ and $\Phi(u)$ is the standard normal integral evaluated at u.

In this case, $U = D_{ij}^2$, $A = 0$, $B = d_{ijw}^2$, $\mu = E[D_{ij}^2]$ and $\sigma^2 = Var[D_{ij}^2]$.

Once the mean and variance of $D_{DP_k}^2$ given $D_{ij}^2 \leq d_{ijw}^2$ is determined, then

$$P[D_{DP_k}^2 > r_{WP_k}^2 U_{ij} \mid D_{ij}^2 \leq d_{ijw}^2] =$$

$$P\left[Z > \frac{r_{WP_k}^2 U_{ij} - E[D_{DP_k}^2 \mid D_{ij}^2 \leq d_{ijw}^2]}{\sqrt{Var[D_{DP_k}^2 \mid D_{ij}^2 \leq d_{ijw}^2]}} \right] \text{ for any } i,j, \quad i \neq j, \text{ where } Z \sim N(0,1).$$

d. Let $p_{no\ shift}(ij \mid w, a_{ij}, agg_{ij})$ be the probability that there is no shift in DGZ (due to preclusion) for the aggregate target formed from units i and j, given that weapon w is used, both units i and j are available, and they can be aggregated with each other.

$$p_{no\ shift}(ij \mid w, a_{ij}, agg_{ij}) \equiv P[\bigcap_k D_{DP_k}^2 > r_{WP_k}^2 \mid D_{ij}^2 \leq d_{ijw}^2]$$
$$\doteq \min_k \left\{ P[D_{DP_k}^2 > r_{WP_k}^2 \mid D_{ij}^2 \leq d_{ijw}^2] \right\}_{k=1}^{np}.$$

This is evaluated as given in the previous paragraph.

e. For the aggregate target DGZ at coordinates $X_{DGZ} = \alpha X_{iL} + (1 - \alpha) X_{jL}$ and $Y_{DGZ} = \alpha Y_{iL} + (1 - \alpha) Y_{jL}$ and the closest preclusion area k away from which the DGZ must shift, calculate the mixing parameter β such that the shifted DGZ has coordinates $X_{DGZ}^s = \beta X_{DGZ} + (1 - \beta) X_{P_k}$ and $Y_{DGZ}^s = \beta Y_{DGZ} + (1 - \beta) Y_{P_k}$. The formulas for computing β were given in Chapter 2.

f. Let $p_{shift}(ij \mid w)$ be the probability that there is a shift in DGZ due to preclusion for the aggregate target formed from units i and j and the shifted DGZ is still within the maximum offset distances d_{iw} and d_{jw}, given that weapon w is used, both units i and j are available, and they can be aggregated with each other. Let $D_{iD^s}^2$ denote the squared distance between the perceived location of unit i and the shifted aggregate DGZ; that is,

$$D_{iD^s}^2 = (X_{DGZ}^s - X_{iL})^2 + (Y_{DGZ}^s - Y_{iL})^2.$$

$$p_{shift}(ij \mid w, a_{ij}, agg_{ij}) =$$
$$P\left(\left\{ \bigcap_k D_{DP_k}^2 > r_{WP_k}^2 U_{ij} \right\} \cap \left\{ \bigcap_k D_{DP_k}^2 > r_{WP_k}^2 \right\}^C \cap \left\{ D_{iD^s}^2 \leq d_{iw}^2 \right\} \cap \left\{ D_{jD^s}^2 \leq d_{jw}^2 \right\} \mid D_{ij}^2 \leq d_{ijw}^2 \right)$$

$$P_{shift}(ij \mid w, a_{ij}, agg_{ij}) =$$

$$\left[P[\, \bigcap_k D^2_{DP_k} > r^2_{WP_k U_{ij}} \mid D^2_{ij} \le d^2_{ij\,w}] - P[\, \bigcap_k D^2_{DP_k} > r^2_{WP_k} \mid D^2_{ij} \le d^2_{ij\,w}]\, \right]$$

$$\cdot\; P[\, D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw} \mid D^2_{ij} \le d^2_{ij\,w}].$$

The latter expression may be evaluated using Bayes' Law:

$$P[\, D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw} \mid D^2_{ij} \le d^2_{ij\,w}]$$

$$= \frac{P[\, D^2_{ij} \le d^2_{ij\,w} \mid D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw}] \cdot P[\, D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw}]}{P[\, D^2_{ij} \le d^2_{ij\,w}]}$$

$$= \frac{1 \cdot P[\, D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw}]}{P[\, D^2_{ij} \le d^2_{ij\,w}]}.$$

$$P_{shift}(ij \mid w, a_{ij}, agg_{ij}) \doteq$$

$$\left[\min_k \left\{ P[\, D^2_{DP_k} > r^2_{WP_k U_{ij}} \mid D^2_{ij} \le d^2_{ij\,w}] \right\}^{np}_{k=1} - \min_k \left\{ P[\, D^2_{DP_k} > r^2_{WP_k} \mid D^2_{ij} \le d^2_{ij\,w}] \right\}^{np}_{k=1} \right]$$

$$\cdot \; \frac{1 \cdot P[\, D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw}]}{P[\, D^2_{ij} \le d^2_{ij\,w}]}.$$

Note that $P_{shift}(ij \mid w, a_{ij}, agg_{ij}) + P_{no\;shift}(ij \mid w, a_{ij}, agg_{ij}) \le P[\, \bigcap_k D^2_{DP_k} > r^2_{WP_k U_{ij}}] = \overline{P}_{prec}(ij \mid w, a_{ij})$ due to the correction factor $P[\, D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw} \mid D^2_{ij} \le d^2_{ij\,w}]$.

Determining the Probability that a Target is Available for Engagement

For every unit i, compute the following:

1. Let $P_{engage}(i, s \mid w)$ be the probability that unit i is available for engagement using weapon w with a DGZ shift, given that weapon w is available to engage the target.

$$P_{engage}(i, s \mid w) = P[\, i \text{ engaged as a single target with DGZ shift} \mid w]$$

$$= P[\, i \text{ available}] \cdot P[\{\, \bigcap_k D_{iP_k} \ge r_{WP_k U_i} \} \cap \{\, \bigcap_k D_{iP_k} \ge r_{WP_k} \}^c \mid w]$$

$$\cdot P[\{\, \bigcup_j \{ (D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\bigcap_k D_{jP_k} \ge r_{WP_k U_j}) \} \}^c \mid w]$$

$$P_{engage}(i, s \mid w) = P_{avail}(i) \cdot P_{shift}(i \mid w, a_i, \overline{agg}_i) \cdot \overline{P}_{aggr}(i \mid w, a_i)$$

2. Let $p_{engage}(i, s, w)$ be the probability that unit i is available for engagement using weapon w with a DGZ shift.

$$p_{engage}(i, s, w) = p_{avail}(i) \cdot p_{round}(w|i) \cdot p_{shift}(i|w, a_i, \overline{agg}_i) \cdot \overline{p}_{aggr}(i|w, a_i)$$

3. Let $p_{engage}(i, \overline{s}|w)$ be the probability that unit i is available for engagement using weapon w without a DGZ shift, given that weapon w is available to engage the target.

$$p_{engage}(i, \overline{s}|w) = P[\, i \text{ engaged as a single target with no DGZ shift } | \, w \,]$$

$$= P[\, i \text{ available }] \cdot P[\, \cap_k D_{iP_k} \geq r_{WP_k} | \, w \,]$$

$$\cdot P[\, \Big\{ \cup_j \{ (D_{ij} \leq d_{ijw}) \cap (j \text{ available }) \cap (\cap_k D_{jP_k} \geq r_{WP_k U_j}) \} \Big\}^c | \, w \,]$$

$$p_{engage}(i, \overline{s}|w) = p_{avail}(i) \cdot p_{no\ shift}(i|w, a_i, \overline{agg}_i) \cdot \overline{p}_{aggr}(i|w, a_i)$$

4. Let $p_{engage}(i, \overline{s}, w)$ be the probability that unit i is available for engagement using weapon w without a DGZ shift.

$$p_{engage}(i, \overline{s}, w) = p_{avail}(i) \cdot p_{round}(w|i) \cdot p_{no\ shift}(i|w, a_i, \overline{agg}_i) \cdot \overline{p}_{aggr}(i|w, a_i)$$

5. Let $p_{engage}(ij, s|w)$ be the probability that the aggregate target formed from units i and j is engaged as an aggregate target with a DGZ shift, given that weapon w is used.

$$p_{engage}(ij, s|w) = P[\, i \text{ engaged as an aggregate target with DGZ shift } | \, w \,]$$

$$= P[\, i \text{ available }] \cdot P[\, j \text{ available }]$$

$$\cdot P[\, \{ \cap_k D_{DP_k} \geq r_{WP_k U_{ij}} \} \cap \{ \cap_k D_{DP_k} \geq r_{WP_k} \}^c | \, D_{ij} \leq d_{ijw} | \, w, a_{ij} \,]$$

$$\cdot P[\, \{ D^2_{iD^s} \leq d^2_{iw} \} \cap \{ D^2_{jD^s} \leq d^2_{jw} \} | \, D_{ij} \leq d_{ijw}, w, a_{ij} \,] \cdot P[\, D_{ij} \leq d_{ijw} | \, w, a_{ij} \,]$$

$$p_{engage}(ij, s|w) = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(ij|w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij|w, a_{ij})$$

6. Let $p_{engage}(ij, s, w)$ be the probability that the aggregate target formed from units i and j is engaged as an aggregate target with a DGZ shift using weapon w.

$$p_{engage}(ij, s, w) = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{round}(w|i, j) \cdot p_{shift}(ij|w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij|w, a_{ij})$$

7. Let $p_{engage}(ij, \overline{s}|w)$ be the probability that the aggregate target formed from units i and j is engaged as an aggregate target with no DGZ shift, given that weapon w is used.

$$p_{engage}(ij, \overline{s}|w) = P[\, i \text{ engaged as an aggregate target with no DGZ shift } | \, w \,]$$

$$= P[i \text{ available}] \cdot P[j \text{ available}] \cdot P[\{ \cap_k D_{DP_k} \geq r_{WP_k} \cap \{ D_{ij} \leq d_{ijw} \} | \, w, a_{ij} \,]$$

$$= P[i \text{ available}] \cdot P[j \text{ available}] \cdot P[\cap_k D_{DP_k} \geq r_{WP_k} | D_{ij} \leq d_{ijw}, w, a_{ij} \,] \cdot P[D_{ij} \leq d_{ijw} | w, a_{ij} \,]$$

$$p_{engage}(ij, \overline{s}|w) = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\ shift}(ij|w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij|w, a_{ij})$$

8. Let $p_{engage}(ij, \overline{s}, w)$ be the probability that the aggregate target formed from units i and j is engaged as an aggregate target without a DGZ shift using weapon w.

$$p_{engage}(ij, \overline{s}, w) = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{round}(w | i, j) \cdot p_{no\ shift}(ij | w, a_{ij}, agg_{ij})$$
$$\cdot\ p_{aggr}(ij | w, a_{ij}).$$

Determining the Probabilities of Conditional Defeat

Let $p_{defeat}(i | w, a_i, \overline{s}, a\overline{g}g_i) \equiv P[$ unit i defeated $|$ weapon w; no DGZ shift; i available; no aggregation $]$ be the probability that unit i can be defeated as a single target, given that weapon w is used, there is no DGZ shift, and unit i is available for fire planning. If unit i is engaged as a single target with no DGZ shift, then the DGZ is located at the perceived target center, with coordinates $X_{DGZ} = X_{iL}$ and $Y_{DGZ} = Y_{iL}$. Using weapon w, unit i is defeated if $(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2 \le d_{iw}^2$. Thus

$$p_{defeat}(i | w, a_i, \overline{s}, a\overline{g}g_i) = P[(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2 \le d_{iw}^2]$$

Let $p_{defeat}(i | w, a_i, s, a\overline{g}g_i) \equiv P[$ unit i defeated $|$ weapon w; DGZ shift; i available; no aggregation$]$ be the probability that unit i can be defeated as a single target, given that weapon w is used, there is a DGZ shift, and unit i is available for fire planning. If unit i is engaged as a single target with a shifted DGZ, then the DGZ is located at the shifted coordinates $X_{DGZ}^s = \beta X_{iL} + (1 - \beta) X_{P_k}$ and $Y_{DGZ}^s = \beta Y_{iL} + (1 - \beta) Y_{P_k}$. Using weapon w, unit i is defeated if $(X_{AGZ}^s - X_i)^2 + (Y_{AGZ}^s - Y_i)^2 \le d_{iw}^2$. Thus

$$p_{defeat}(i | w, a_i, s, a\overline{g}g_i) = P[(X_{AGZ}^s - X_i)^2 + (Y_{AGZ}^s - Y_i)^2 \le d_{iw}^2].$$

Let $p_{defeat}(ij | w, a_{ij}, \overline{s}, agg_{ij}) \equiv P[$ units i and j defeated as aggregate target $|$ weapon w; no DGZ shift; i,j available $]$ be the probability that units i and j can be defeated as an aggregate target, given that weapon w is used, there is no DGZ shift, and units i and j are available for fire planning. If units i and j are engaged as an aggregate target with no DGZ shift, then the DGZ is located along a line segment connecting the perceived target centers, with coordinates $X_{DGZ} = \alpha X_{iL} + (1 - \alpha) X_{jL}$ and $Y_{DGZ} = \alpha Y_{iL} + (1 - \alpha) Y_{jL}$. Using weapon w, unit i is defeated if $(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2 \le d_{iw}^2$ and unit j is defeated if $(X_{AGZ} - X_j)^2 + (Y_{AGZ} - Y_j)^2 \le d_{jw}^2$. Thus

$$p_{defeat}(ij | w, a_{ij}, \overline{s}, agg_{ij})$$
$$= P[\ \{(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2 \le d_{iw}^2\} \cap \{(X_{AGZ} - X_j)^2 + (Y_{AGZ} - Y_j)^2 \le d_{jw}^2\}\ ]$$

Let $P_{defeat}(ij \mid w, a_{ij}, s, agg_{ij}) \equiv P[$ units i and j defeated as aggregate target | weapon w; DGZ shift; i,j available] be the probability that units i and j can be defeated as an aggregate target, given that weapon w is used, there is a DGZ shift, and units i and j are available for fire planning. If units i and j are engaged as an aggregate target with a DGZ shift due to preclusion, then the DGZ is shifted from the point located along a line segment connecting the perceived target centers, with shifted coordinates $X^s_{DGZ} = \beta X_{DGZ} + (1 - \beta) X_{P_k}$ and $Y^s_{DGZ} = \beta Y_{DGZ} + (1 - \beta) Y_{P_k}$, where the unshifted DGZ had coordinates $X_{DGZ} = \alpha X_{iL} + (1-\alpha) X_{jL}$ and $Y_{DGZ} = \alpha Y_{iL} + (1-\alpha) Y_{jL}$. Using weapon w, unit i is defeated if $(X^s_{AGZ} - X_i)^2 + (Y^s_{AGZ} - Y_i)^2 \leq d^2_{iw}$ and unit j is defeated if $(X^s_{AGZ} - X_j)^2 + (Y^s_{AGZ} - Y_j)^2 \leq d^2_{jw}$. Thus

$$P_{defeat}(ij \mid w, a_{ij}, s, agg_{ij})$$
$$= P[\{(X^s_{AGZ}-X_i)^2+(Y^s_{AGZ}-Y_i)^2 \leq d^2_{iw}\} \cap \{(X^s_{AGZ}-X_j)^2+(Y^s_{AGZ}-Y_j)^2 \leq d^2_{jw}\}]$$

Let $P_{defeat}(i \mid w, a_{ij}, \bar{s}, agg_{ij}) \equiv P[$ unit i defeated as aggregate target | weapon w; no DGZ shift; i,j available] be the probability that unit i can be defeated as an aggregate target, given that weapon w is used, there is no DGZ shift, and units i and j are available for fire planning. If units i and j are engaged as an aggregate target with no DGZ shift, then the DGZ is located along a line segment connecting the perceived target centers, with coordinates $X_{DGZ} = \alpha X_{iL} + (1-\alpha) X_{jL}$ and $Y_{DGZ} = \alpha Y_{iL} + (1-\alpha) Y_{jL}$. Using weapon w, unit i is defeated if $(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2 \leq d^2_{iw}$. Thus

$$P_{defeat}(i \mid w, a_{ij}, \bar{s}, agg_{ij}) = P[(X_{AGZ} - X_i)^2+(Y_{AGZ} - Y_i)^2 \leq d^2_{iw}]$$

Let $P_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) \equiv P[$ unit i defeated as aggregate target | weapon w; DGZ shift; i,j available] be the probability that unit i can be defeated as an aggregate target, given that weapon w is used, there is a DGZ shift, and units i and j are available for fire planning. If units i and j are engaged as an aggregate target with a DGZ shift due to preclusion, then the DGZ is shifted from the point located along a line segment connecting the perceived target centers, with shifted coordinates $X^s_{DGZ} = \beta X_{DGZ} + (1 - \beta) X_{P_k}$ and $Y^s_{DGZ} = \beta Y_{DGZ} + (1 - \beta) Y_{P_k}$, where the unshifted DGZ had coordinates $X_{DGZ} = \alpha X_{iL} + (1-\alpha) X_{jL}$ and $Y_{DGZ} = \alpha Y_{iL} + (1-\alpha) Y_{jL}$. Using weapon w, unit i is defeated if $(X^s_{AGZ} - X_i)^2 + (Y^s_{AGZ} - Y_i)^2 \leq d^2_{iw}$. Thus

$$P_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) = P[(X^s_{AGZ} - X_i)^2+(Y^s_{AGZ} - Y_i)^2 \leq d^2_{iw}$$

Determining the Probability of Defeat

For unit i, there are three exhaustive and mutually exclusive outcomes:

1. Unit i is not engaged
2. Unit i is engaged as a single target
3. Unit i is engaged as an aggregate target

We will consider each in turn.

1. Unit i not engaged. This occurs when:

a. Unit i not available ($\overline{p}_{avail}(i)$), *or*

b. Unit i is precluded from fire ($\underset{w}{\cup} \underset{k}{\cup} D_{iP_k} < r_{WP_k U_i}$), *or*

c. Round w is not available, for any weapon w not precluded

$$(\underset{w}{\cup} \left\{ \underset{k}{\cap} D_{iP_k} \geq r_{WP_k U_i} \cap w \text{ not available} \right\})$$

Thus, unit i is not engaged when $\underset{w}{\cup} \left\{ \underset{k}{\cup} D_{iP_k} < r_{WP_k U_i} \cup w \text{ not available} \right\} \cup \left\{ i \text{ not available} \right\}$

2. Unit i engaged as a single target. This occurs when:

a. Unit i available ($p_{avail}(i)$), *and*

b. Unit i is not precluded from fire ($\underset{w}{\cup} \underset{k}{\cap} D_{iP_k} \geq r_{WP_k U_i}$), *and*

c. Round w is available for any weapon w not precluded

$$(\underset{w}{\cup} \left\{ \underset{k}{\cap} D_{iP_k} \geq r_{WP_k U_i} \cap w \text{ available} \right\})$$

d. Unit i cannot be engaged as an aggregate target with any unit j. This occurs when:

(1) $D_{ij} > d_{ijw}$, *or*

(2) Unit j is not available ($\overline{p}_{avail}(j)$), *or*

(3) Unit j is precluded from fire ($\underset{w}{\cup} \underset{k}{\cup} D_{jP_k} < r_{WP_k U_j}$)

Thus unit i *can* be engaged as an aggregate target with any unit j, given unit i available, i not precluded and w available, when

$$(D_{ij} \leq d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \geq r_{WP_k U_j}),$$

thus unit i *cannot* be engaged as an aggregate target with any unit j, given unit i available, i not precluded and w available, when

$$\left\{ \underset{j}{\cup} \left\{ (D_{ij} \leq d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \geq r_{WP_k U_j}) \right\} \right\}^c.$$

Let us denote $(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2$ as D_{Ai}. Given a weapon w, unit i is defeated as a single target when

$$(D_{Ai} \leq d_{iw} \cap i \text{ engaged as a single target} \mid w)$$

with probability

$$P[\ D_{A_i} \le d_{iw}\ |\ i \text{ engaged as a single target},\ w\ ] \cdot P[\ i \text{ engaged as a single target}\ |\ w\ ]$$

which equals

$$P[\ D_{A_i} \le d_{iw}\ |\ \text{DGZ shift},\ i \text{ single target},\ w\ ] \cdot P[\ \text{DGZ shift},\ i \text{ single target}\ |\ w\ ]$$
$$+ P[\ D_{A_i} \le d_{iw}\ |\ \text{no DGZ shift},\ i \text{ single target},\ w\ ] \cdot P[\ \text{no DGZ shift},\ i \text{ single target}\ |\ w\ ]$$

Thus, $P[\ i \text{ engaged as a single target with DGZ shift}\ |\ w\ ] =$

$$P[\ \{i \text{ available}\} \cap \{\underset{k}{\cap} D_{iP_k} \ge {}^r W P_k U_i\} \cap \{\underset{k}{\cup} D_{iP_k} < {}^r W P_k\}$$
$$\cap\Big\{\ \underset{j}{\cup}\ \{\ (D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge {}^r W P_k U_j)\ \}\ \Big\}^c\ |\ w\ ]$$
$$= P[\ \{i \text{ available}\} \cap \Big\{ \underset{k}{\cap}\{\ D_{iP_k} \ge {}^r W P_k U_i\} \cap \{\underset{k}{\cup} D_{iP_k} < {}^r W P_k\}\ \Big\}$$
$$\cap\Big\{\ \underset{j}{\cup}\ \{\ (D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge {}^r W P_k U_j)\ \}\ \Big\}^c\ |\ w\ ]$$

By assumption, the probabilities that i is available, that it is not precluded but the DGZ is shifted, and that it cannot be aggregated with any unit j are independent given w. Thus

$$P[\ i \text{ engaged as a single target with DGZ shift}\ |\ w\ ]$$

$$= P[\ i \text{ available}\ ] \cdot P[\underset{k}{\cap}\{\ D_{iP_k} \ge {}^r W P_k U_i\} \cap \{\underset{k}{\cup} D_{iP_k} < {}^r W P_k\}\ |\ w,\ a_i\ ]$$
$$\cdot\ P[\ \Big\{\ \underset{j}{\cup}\ \{\ (D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge {}^r W P_k U_j)\ \}\ \Big\}^c\ |\ w,\ a_i\ ]$$
$$= p_{avail}(i) \cdot p_{shift}(i\,|\,w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i\,|\,w, a_i)$$

Similarly, $P[\ i \text{ engaged as a single target with no DGZ shift}\ |\ w\ ] =$

$$P[\ \{i \text{ available}\} \cap \{\underset{k}{\cap} D_{iP_k} \ge {}^r W P_k U_i\} \cap \{\underset{k}{\cap} D_{iP_k} \ge {}^r W P_k\}$$
$$\cap\Big\{\ \underset{j}{\cup}\ \{\ (D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge {}^r W P_k U_j)\ \}\ \Big\}^c\ |\ w,\ a_i\ ]$$
$$= P[\ \{i \text{ available}\} \cap \Big\{ \underset{k}{\cap}\{\ D_{iP_k} \ge {}^r W P_k\}$$
$$\cap\Big\{\ \underset{j}{\cup}\ \{\ (D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge {}^r W P_k U_j)\ \}\ \Big\}^c\ |\ w,\ a_i\ ]$$

By assumption, the probabilities that i is available, that it is not precluded and the DGZ is not shifted, and that it cannot be aggregated with any unit j, are independent given w. Thus

$$P[\ i \text{ engaged as a single target with no DGZ shift}\ |\ w\ ]$$

$$= P[\ i \text{ available}\ ] \cdot P[\underset{k}{\cap} D_{iP_k} \ge {}^r W P_k\ |\ w\ ]$$
$$\cdot\ P[\ \Big\{\ \underset{j}{\cup}\ \{\ (D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \ge {}^r W P_k U_j)\ \}\ \Big\}^c\ |\ w,\ a_i\ ]$$
$$= p_{avail}(i) \cdot p_{no\ shift}(i\,|\,w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i\,|\,w, a_i)$$

3. Unit i engaged as an aggregate target. This occurs when:

a. Unit i available ($p_{avail}(i)$), and

b. Unit i is not precluded from fire ($\underset{w}{\cup} \underset{k}{\cap} D_{iP_k} \geq r_{WP_k U_i}$), and

c. Round w is available for any weapon w not precluded, and

$$(\underset{w}{\cup} \left\{ \underset{k}{\cap} D_{iP_k} \geq r_{WP_k U_i} \cap w \text{ available} \right\})$$

d. Unit i can be engaged as an aggregate target with some unit j. This occurs when:

(1) $D_{ij} \leq d_{ijw}$, and

(2) Unit j is available ($p_{avail}(j)$), and

(3) Unit j is not precluded from fire ($\underset{w}{\cup} \underset{k}{\cap} D_{iP_k} \geq r_{WP_k U_i}$).

Thus unit i *can* be engaged as an aggregate target with any unit j, given unit i available, i not precluded and w available, when ($D_{ij} \leq d_{ijw}$) $\cap$ (j not available) $\cap$ ($\underset{k}{\cap} D_{iP_k} \geq r_{WP_k U_i}$), and

e. The aggregate DGZ is not precluded from fire ($\underset{w}{\cup} \underset{k}{\cap} D_{DP_k} \geq r_{WP_k U_{ij}}$), and

f. If the aggregate DGZ is shifted, the shifted DGZ must still achieve coverage over units i and j, when ($D^2_{iD^s} \leq d^2_{iw}$) $\cap$ ($D^2_{jD^s} \leq d^2_{jw}$)

Recall that $(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2 = D_{Ai}$. Given a weapon w, unit i is defeated as an aggregate target when

($D_{Ai} \leq d_{iw} \cap ij$ engaged as an aggregate target $| w$)

with probability

$P[D_{Ai} \leq d_{iw} | ij$ engaged as an aggregate target, $w] \cdot P[ij$ engaged as an aggregate target $| w]$

which equals

$P[D_{Ai} \leq d_{iw} |$ DGZ shift, ij aggregate target, $w] \cdot P[$ DGZ shift, ij aggregate target $| w]$

$+ P[D_{Ai} \leq d_{iw} |$ no DGZ shift, ij aggregate target, $w] \cdot P[$ no DGZ shift, ij aggregate target $| w]$

Thus, $P[ij$ engaged as an aggregate target with DGZ shift $| w] =$

$P[\{ ij \text{ available} \} \cap \{ \underset{k}{\cap} D_{iP_k} \geq r_{WP_k U_i} \}$

$\cap \{ \underset{k}{\cap} D_{DP_k} \geq r_{WP_k U_{ij}} \} \cap \{ \underset{k}{\cup} D_{DP_k} < r_{WP_k} \} \cap \{ D^2_{iD^s} \leq d^2_{iw} \} \cap \{ D^2_{jD^s} \leq d^2_{jw} \}$

$\cap \left\{ (D_{ij} \leq d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \geq r_{WP_k U_j}) \right\} | w]$

Since $\{ \underset{k}{\cap} D_{DP_k} \geq r_{WP_k U_{ij}} \} \cap \{ D^2_{iD^s} \leq d^2_{iw} \} \cap \{ D^2_{jD^s} \leq d^2_{jw} \}$ cannot occur unless $\{ \underset{k}{\cap} D_{iP_k} \geq r_{WP_k U_i} \} \cap \{ \underset{k}{\cap} D_{jP_k} \geq r_{WP_k U_j} \}$ occurs, we can drop the latter two terms.

$= P[\{ i \text{ available} \} \cap \{ \underset{k}{\cap} D_{DP_k} \geq r_{WP_k U_{ij}} \} \cap \{ \underset{k}{\cup} D_{DP_k} < r_{WP_k} \} \} \cap \{ D^2_{iD^s} \leq d^2_{iw} \}$

$\cap \{ D^2_{jD^s} \leq d^2_{jw} \} \cap \{ j \text{ available} \} \cap \{ D_{ij} \leq d_{ijw} \} | w]$

By assumption, the probabilities that i is available and not precluded, that j is available and not precluded, and the probability that the aggregate DGZ is shifted and that i can be aggregated with unit j, are independent given w. Thus

P[i engaged as an aggregate target with DGZ shift $| w$]

$= $ P[i available] $\cdot$ P[j available] $\cdot$

$$P[\{\underset{k}{\cap}D_{DP_k} \geq {}^r W P_k U_{ij}\} \cap \{D^2_{iD^s} \leq d^2_{iw}\} \cap \{D^2_{jD^s} \leq d^2_{jw}\} \cap \{\underset{k}{\cup}D_{DP_k} < {}^r W P_k\} \cap \{D_{ij} \leq d_{ijw}\} \mid w, a_{ij}]$$

$= $ P[i available] $\cdot$ P[j available]

$\cdot$ P[$\{\underset{k}{\cap}D_{DP_k} \geq {}^r W P_k U_{ij}\} \cap \{\underset{k}{\cap}D_{DP_k} \geq {}^r W P_k\}^c \mid D_{ij} \leq d_{ijw} \mid w, a_{ij}$]

$\cdot$ P[$\{D^2_{iD^s} \leq d^2_{iw}\} \cap \{D^2_{jD^s} \leq d^2_{jw}\} \mid D_{ij} \leq d_{ijw} \mid w$] $\cdot$ P[$D_{ij} \leq d_{ijw} \mid w, a_{ij}$]

$= $ $p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij})$

Similarly, P[i engaged as an aggregate target with no DGZ shift $| w$] $=$

P[$\{i$ available$\} \cap \{\underset{k}{\cap}D_{iP_k} \geq {}^r W P_k U_i\} \cap \{\underset{k}{\cap}D_{DP_k} \geq {}^r W P_k\}$

$\cap \Big\{ (D_{ij} \leq d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \geq {}^r W P_k U_j) \Big\} \mid w$]

Since $\{\underset{k}{\cap}D_{DP_k} \geq {}^r W P_k\}$ cannot occur unless $\{\underset{k}{\cap}D_{iP_k} \geq {}^r W P_k U_i\} \cap \{\underset{k}{\cap}D_{jP_k} \geq {}^r W P_k U_j\}$ occurs. we can drop the latter two terms.

$= $ P[$\{i$ available$\} \cap \{j$ available$\} \cap \{\underset{k}{\cap}D_{DP_k} \geq {}^r W P_k\} \cap \Big\{ (D_{ij} \leq d_{ijw}) \mid w$]

By assumption, the probabilities that i is available and not precluded, that j is available and not precluded, and the probability that the aggregate DGZ is not shifted and that i can be aggregated with unit j, are independent given w. Thus

P[ij engaged as an aggregate target with no DGZ shift $| w$]

$= $ P[i available] $\cdot$ P[j available] $\cdot$ P[$\{\underset{k}{\cap} D_{DP_k} \geq {}^r W P_k\} \cap \{D_{ij} \leq d_{ijw}\} \mid w, a_{ij}$]

$= $ P[i available] $\cdot$ P[j available] $\cdot$ P[$\underset{k}{\cap} D_{DP_k} \geq {}^r W P_k \mid D_{ij} \leq d_{ijw}, w$] $\cdot$ P[$D_{ij} \leq d_{ijw} \mid w, a_{ij}$]

$= $ $p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\ shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij})$

4. Summary.

a. Given a weapon w, unit i is defeated as a single target with probability

P[$D_{Ai} \leq d_{iw} \mid$ DGZ shift, i single target, w] $\cdot$ P[DGZ shift, i single target $| w$]

$\quad + $ P[$D_{Ai} \leq d_{iw} \mid$ no DGZ shift, i single target, w] $\cdot$ P[no DGZ shift, i single target $| w$]

$= $ $p_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) \cdot p_{avail}(i) \cdot p_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i)$

$\quad + $ $p_{defeat}(i \mid w, a_i, \overline{s}, a\overline{g}g_i) \cdot p_{avail}(i) \cdot p_{no\ shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i)$

b. Given a weapon w, unit i is defeated as an aggregate target with unit j with probability

$P[D_{A_i} \leq d_{iw} \mid \text{DGZ shift, } ij \text{ aggregate target, } w] \cdot P[\text{ DGZ shift, } ij \text{ aggregate target } \mid w]$

$+ P[D_{A_i} \leq d_{iw} \mid \text{no DGZ shift, } ij \text{ aggregate target, } w] \cdot P[\text{ no DGZ shift, } ij \text{ aggregate target } \mid w]$

$= p_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) \cdot p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij})$

$+ p_{defeat}(i \mid w, a_{ij}, \overline{s}, agg_{ij}) \cdot p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\ shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij})$

c. If we select j such that j solves $\max_{j} \left\{ p_{aggr}(ij \mid w, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j \mid w, a_j, \overline{agg}_j) \right\}_{\substack{j=1 \\ j \neq i}}^{m}$.

then $p_{engage}(i \mid w) = p_{avail}(i) \cdot \overline{p}_{prec}(i \mid w, a_i, \overline{agg}_i) \cdot [1 - p_{avail}(j) \cdot \overline{p}_{prec}(j \mid w, a_j, \overline{agg}_j)$

$\cdot p_{aggr}(ij \mid w, a_{ij})]$. We note that the event with probability $\overline{p}_{prec}(ij \mid w, a_{ij}, agg_j)$ implies the events

that have probabilities $\overline{p}_{prec}(i \mid w, a_i, \overline{agg}_i)$ and $\overline{p}_{prec}(j \mid w, a_j, \overline{agg}_j)$ (i.e., the lack of preclusion of

the aggregate target ij implies the lack of preclusion of the individual targets i and j). If we

approximate

$p_{engage}(ij \mid w) = p_{avail}(i) \cdot p_{avail}(j) \cdot [p_{no\ shift}(ij \mid w, a_i, agg_{ij}) + p_{shift}(ij \mid w, a_i, agg_{ij})] \cdot$

$p_{aggr}(ij \mid w, a_{ij})$ with

$\hat{p}_{engage}(ij \mid w) = p_{avail}(i) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(i \mid w, a_i, \overline{agg}_i) \cdot \overline{p}_{prec}(j \mid w, a_j, \overline{agg}_j) \cdot$

$p_{aggr}(ij \mid w, a_{ij})$, then

$p_{engage}(i \mid w) + \hat{p}_{engage}(ij \mid w) = p_{avail}(i) \cdot \overline{p}_{prec}(i \mid w, a_i, \overline{agg}_i)$.

In other words, the probability of engagement of unit i as a single target or as an aggregate target (given w) is approximately the probability that the unit is available and not precluded. The actual probability of engagement given w will be slightly less, due to additional requirements for an aggregate target to be engaged (the aggregated DGZ must meet both preclusion and minimum coverage requirements).

Allocating Weapons to Targets

We are interested in the probability $p_{round}(w \mid a_i)$, the probability that weapon w is available to engage target unit i, given that unit i is available, and the probability $p_{round}(w \mid a_{ij})$, the probability that weapon w is available to engage target units i and j (as an aggregate target), given that units i and j are available.

Because the number of potential target units is much larger than the supply of available weapons, the probability that a weapon w can be allocated to an acquired unit i depends not only on the unit's priority as a target, but also on the actual number of other target units of equal or higher

priority that are available as potential targets. The weapon allocations, therefore, are highly dependent between units. We can estimate the marginal probabilities properly, taking into account this dependence, using either of the two approaches given below.

Given any realization of the target unit set available for planning from all units i and aggregate pairs i,j, it is possible to determine an assignment of weapons w such that the greatest number of high priority targets are engaged. This allocation can be accomplished using any doctrinally defined allocation scheme or a standard "assignment" type linear programming (LP) code. Unfortunately, there are 2^{m^2} possible combinations of m^2 binary variables representing the defeat / failure to defeat each unit, so it is not feasible to determine all possible weapon allocations for each potential target unit for large m.

An estimate of $p_{round}(w|a_i)$ can be formed by drawing binary random variables using the probabilities $p_{engage}(ij|w) = p_{engage}(ij,\overline{s}|w) + p_{engage}(ij,s|w)$, for all unit pairs $i,j = 1, \ldots, m$. For those units i,j where the binary variable for engagement as an aggregate target is zero, draw another binary variable against $p_{engage}(k|w) = p_{engage}(k,\overline{s}|w) + p_{engage}(k,s|w)$, for $k = i, j$. Assign available weapons within range (based on some allocation scheme such as top-down or bottom-up yields) to the targets with positive binary variables, continuing until all available weapons are assigned. This forms an estimate of the joint probability P[unit i is available for engagement using weapon w and weapon w is available]. If we repeat this process n times, we average out $p_{engage}(i|w)$ and $p_{engage}(ij|w)$ and form estimates $\hat{p}_{round}(w|a_i)$ and $\hat{p}_{round}(w|a_{ij})$ of $p_{round}(w|a_i)$ and $p_{round}(w|a_{ij})$ respectively.

In developing a Monte Carlo estimate of the probability (by weapon type) that a round is available for a given target unit, it is necessary to draw against the probability that the target unit is available for fire. Let i_s denote the event that unit i is engaged as a single target and let i_a denote the event that unit i is engaged as an aggregate target. P[$i_s|w$] = P[{ i available } $\cap$ { $\underset{k}{\cap} D_{iP_k} \geq$ $r_{WP_k U_i}$ } $\cap$ { $(D_{ij^*} \leq d_{ij^* w}) \cap (j^*$ available $) \cap (\underset{k}{\cap} D_{j^* P_k} \geq r_{WP_k U_{j^*}})$ }^{c} $|w$] = $p_{engage}(i,\overline{s}|w)$ + $p_{engage}(i,s|w)$ = $p_{engage}(i|w)$, where j^* maximizes { $p_{aggr}(ij|w,a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j|w,a_j,a\overline{g}g_j)$ }. Similarly, P[$i_a|w$] = P[{ i available } $\cap$ { $\underset{k}{\cap} D_{DP_k} \geq r_{WP_k U_{ij^*}}$ } $\cap$ { $D^2_{iD^s} \leq d^2_{iw}$ } $\cap$ { $D^2_{j^* D^s} \leq d^2_{j^* w}$ } $\cap$ { j^* available } $\cap$ { $D_{ij^*} \leq d_{ij^* w}$ } $|w$] = $p_{engage}(ij,\overline{s}|w)$ + $p_{engage}(ij,s|w)$ = $p_{engage}(ij|w)$. P[$i_s \cup i_a |w$] = P[$i_s|w$] + P[$i_a|w$] $-$ P[$i_s \cap i_a|w$].

Note that

$$P[\, i_s \cap i_a \mid w \,] = P\Big[\Big\{ \{ i \text{ available} \} \cap \{ \cap_k D_{i P_k} \geq \, ^r W P_k \, U_i \} \cap \Big\{ (D_{ij^*} \leq d_{ij^*w}) \cap (j^* \text{ available})$$

$$\cap \, (\cap_k D_{j^* F_k} \geq \, ^r W P_k \, U_{j^*}) \Big\}^c \Big\} \cap \Big\{ \{ i \text{ available} \} \cap \{ \cap_k D_{D P_k} \geq \, ^r W P_k \, U_{ij^*} \}$$

$$\cap \, \{ D^2_{iD^s} \leq d^2_{iw} \} \cap \{ D^2_{j^* D^s} \leq d^2_{j^*w} \} \cap \{ j^* \text{ available} \} \cap \{ D_{ij^*} \leq d_{ij^*w} \} \Big\} \mid w \,]$$

which includes the terms $\Big\{ \{ D_{ij^*} \leq d_{ij^*w} \} \cap \{ j^* \text{ available} \} \cap \{ \cap_k D_{j^* P_k} \geq \, ^r W P_k \, U_{j^*} \} \Big\}^c \cap$

$\Big\{ \{ \cap_k D_{D P_k} \geq \, ^r W P_k \, U_{ij^*} \} \cap \{ j^* \text{ available} \} \cap \{ D_{ij^*} \leq d_{ij^*w} \} \Big\}$. If we continue to assume that

$\{ \cap_k D_{D P_k} \geq \, ^r W P_k \, U_{ij^*} \}$ implies $\{ \cap_k D_{j^* P_k} \geq \, ^r W P_k \, U_{j^*} \}$, then we have a probability statement of

the form $P[\, A^c \cap A \,]$, which equals zero. Thus $P[\, i_s \cup i_a \mid w \,] = P[\, i_s \mid w \,] + P[\, i_a \mid w \,] =$

$p_{engage}(i \mid w) + p_{engage}(ij \mid w)$. In practice, we always determine a single j for each i such that j

maximizes $\Big\{ \overline{p}_{aggr}(ij \mid w, a_{ij}) \cdot p_{avail}(j \mid w) \cdot \overline{p}_{prec}(j \mid w, a_j, a\overline{g}g_j) \Big\}$, so there is only one target unit j

to consider as an aggregate target for each unit i.

To develop a Monte Carlo estimate of the probability (by weapon type) that a round is

available for a given target unit, we begin by generating realizations of single and aggregate target

sets. To do this, we draw against the probability that the target unit is available for fire as follows:

ALGORITHM:

1. For each target unit i, $i = 1, \ldots, m$,

2.   Draw $U_i \sim$ Uniform(0,1).

3.   For each weapon type w, $w = 1, \ldots, nw$,

4.   Using j maximizing $\Big\{ \overline{p}_{aggr}(ij \mid w, a_{ij}) \cdot p_{avail}(j \mid w) \cdot \overline{p}_{prec}(j \mid w, a_j, a\overline{g}g_j) \Big\}$,

         if $U_i < p_{engage}(i \mid w) + p_{engage}(ij \mid w)$, let $B(i, w) = 1$

5.   Also if $U_i < p_{engage}(i \mid w)$, let $A(i, w) = 1$

6.   End if

7.   End loop on w

8. End loop on i

9. The available target set is generated as follows:

     If $B(i, w) = 0$, target unit i is not available for fire

     If $A(i, w) = 1$, target unit i is available for fire as a single target

     If $A(i, w) = 0$ and $B(i, w) = 1$, target unit i is available for fire as an aggregate target

Some alternative approaches involving bounding the probability that a weapon w can be used to

engage unit i exist but may not yield sufficiently tight bounds for the purpose of estimation.

Formulas for the Probability of Defeat, Given Weapon W

Let $p_{defeat}(i|w) \equiv P[$ unit i defeated $|$ weapon $w]$ be the probability that unit i is available and can be defeated, given that weapon w is used. Thus we are averaging over the probability that unit i is available, that it is engaged as a single or aggregate target, and that the DGZ is or is not shifted.

$p_{defeat}(i|w)$

$= P[$ unit i defeated as a single target $|w] + P[$ unit i defeated as an aggregate target $|w]$

$= P[D_{A_i} \le d_{iw} |$ DGZ shift, i single target, $w] \cdot P[$ DGZ shift, i single target $|w]$

$+ P[D_{A_i} \le d_{iw} |$ no DGZ shift, i single target, $w] \cdot P[$ no DGZ shift, i single target $|w]$

$+ P[D_{A_i} \le d_{iw} |$ DGZ shift, ij aggregate target, $w] \cdot P[$ DGZ shift, ij aggregate target $|w]$

$+ P[D_{A_i} \le d_{iw} |$ no DGZ shift, ij aggregate target, $w] \cdot P[$ no DGZ shift, ij aggregate target $|w]$

$p_{defeat}(i|w)$

$= p_{defeat}(i|w, a_i, s, a\overline{g}g_i) \cdot p_{avail}(i) \cdot p_{shift}(i|w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i|w, a_i)$

$+ p_{defeat}(i|w, a_i, \overline{s}, a\overline{g}g_i) \cdot p_{avail}(i) \cdot p_{no\ shift}(i|w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i|w, a_i)$

$+ p_{defeat}(i|w, a_{ij}, s, agg_{ij}) \cdot p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(ij|w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij|w, a_{ij})$

$+ p_{defeat}(i|w, a_{ij}, \overline{s}, agg_{ij}) \cdot p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\ shift}(ij|w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij|w, a_{ij})$

$p_{defeat}(i|w)$

$= p_{defeat}(i|w, a_i, s, a\overline{g}g_i) \cdot p_{engage}(i, s|w) + p_{defeat}(i|w, a_i, \overline{s}, a\overline{g}g_i) \cdot p_{engage}(i, \overline{s}|w)$

$+ p_{defeat}(i|w, a_{ij}, s, agg_{ij}) \cdot p_{engage}(ij, s|w) + p_{defeat}(i|w, a_{ij}, \overline{s}, agg_{ij}) \cdot p_{engage}(ij, \overline{s}|w)$

Estimating the Probability that a Unit is Defeated

Given the estimate of the probability that a unit i is available for engagement by weapon w, $p_{avail}(i|w)$, and the probability that weapon w will be allocated to that target, $p_{round}(w|i)$, it is possible to estimate the probability that a unit is defeated. When the number of rounds available is constrained, the probabilities of defeat are dependent between the various units, with a joint dependence that depends on the total numbers of higher priority units being defeated (that is, the probability that unit i is defeated given that many higher priority units are defeated is less than the marginal probability that unit i is defeated). However, this generally becomes a problem only when we look at extremes, where almost all or almost none of the units are available for engagement. The

effect of this dependence will be to alter the joint probability of many events across the theater, but it should not change the determination of the modes within partitions of the sample space (Chapter 1), nor will it be important when examining smaller subsets of units. Caution in interpreting the joint results of a laydown should be exercised when using this approach as a detailed model surrogate.

If we consider the probabilities independently between units, we can estimate the probability that any unit i can be defeated as follows. Let $p_{defeat}(i, w) \equiv P[$unit i defeated using weapon $w]$ be the joint probability that unit i is available and can be defeated using weapon w. In this case, we remove the conditioning on weapon w.

$$
\begin{aligned}
p_{defeat}&(i, w) \\
&= \Big\{ p_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) \cdot p_{avail}(i) \cdot p_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) \\
&\quad + p_{defeat}(i \mid w, a_i, \overline{s}, a\overline{g}g_i) \cdot p_{avail}(i) \cdot p_{no\ shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) \Big\} \\
&\quad \cdot p_{round}(w \mid a_i) \\
&\quad + \Big\{ p_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) \cdot p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij}) \\
&\quad + p_{defeat}(i \mid w, a_{ij}, \overline{s}, agg_{ij}) \cdot p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\ shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij}) \Big\} \\
&\quad \cdot p_{round}(w \mid a_{ij})
\end{aligned}
$$

$$
\begin{aligned}
p_{defeat}&(i, w) \\
&= p_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) \cdot p_{engage}(i, s, w) + p_{defeat}(i \mid w, a_i, \overline{s}, a\overline{g}g_i) \cdot p_{engage}(i, \overline{s}, w) \\
&\quad + p_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) \cdot p_{engage}(ij, s, w) + p_{defeat}(i \mid w, a_{ij}, \overline{s}, agg_{ij}) \cdot p_{engage}(ij, \overline{s}, w)
\end{aligned}
$$

The probability that unit i can be defeated, denoted as $p_{defeat}(i)$, is simply

$$
p_{defeat}(i) = \sum_w P[\text{ Unit } i \text{ defeated using weapon } w] \text{ or}
$$

$$
p_{defeat}(i) = \sum_w p_{defeat}(i, w).
$$

Bonus Effects

If desired, the probability that a unit may be defeated can be adjusted to include bonus effects, that is, the probability that unit i is defeated due to a burst aimed at unit j or aggregate target jk.

Let (X_{AGZ_j}, Y_{AGZ_j}) denote the coordinates of the AGZ for the weapon fired at unit j, and let $D^2_{i\,AGZ_j} \equiv (X_{AGZ_j} - X_i)^2 + (Y_{AGZ_j} - Y_i)^2$. The coordinates (X_{AGZ_j}, Y_{AGZ_j}) will depend upon whether j was engaged as a single or aggregate target with or without a DGZ shift. Thus

$$P_{defeat}(\imath \mid AGZ_{\jmath},\, w) = P[\, D^2_{\imath AGZ_{\jmath}} \leq d^2_{\imath w} \mid AGZ_{\jmath},\, w\,] \text{ and}$$

$$\begin{aligned}
P_{defeat}(\imath,\, AGZ_{\jmath},\, w) = {}& P[\, D^2_{\imath AGZ_{\jmath}} \leq d^2_{\imath w} \mid w,\, a_{\jmath},\, s,\, \overline{agg}_{\jmath}\,] \cdot P_{engage}(\jmath, s, w) \\
& + P[\, D^2_{\imath AGZ_{\jmath}} \leq d^2_{\imath w} \mid w,\, a_{\jmath},\, \overline{s},\, \overline{agg}_{\jmath}\,] \cdot P_{engage}(\jmath, \overline{s}, w) \\
& + P[\, D^2_{\imath AGZ_{\jmath k}} \leq d^2_{\imath w} \mid w,\, a_{\jmath},\, s,\, agg_{\jmath k}\,] \cdot P_{engage}(\jmath k, s, w) \\
& + P[\, D^2_{\imath AGZ_{\jmath k}} \leq d^2_{\imath w} \mid w,\, a_{\jmath},\, \overline{s},\, a_{\jmath} g_{\jmath k}\,] \cdot P_{engage}(\jmath k, \overline{s}, w)
\end{aligned}$$

The total probability of defeat will be

$$P_{defeat}(\imath) = \sum_{w=1}^{nw} \left[P_{defeat}(\imath,\, w) + \sum_{\substack{\jmath=1 \\ \jmath \neq \imath}}^{m} P_{defeat}(\imath,\, AGZ_{\jmath},\, w) \right].$$

It is anticipated that, although bonus damage may be realized, the probability that a unit is defeated due to bonus damage will be very small, and normally may safely be ignored.

Estimating the Sample Space of All Possible Outcomes of a Theater Nuclear Exchange

If we regard the outcome of a theater nuclear exchange upon each unit from the requirements of the defeat criteria, we can define the outcome of the exchange on unit $\imath$ as a binary variable $O_{\imath}$, where $O_{\imath} = 1$ if the unit is defeated; 0 otherwise. Given the assumption that the outcome is independent between units, the outcome of any exchange is simply a set of 0's and 1's with the probability that any $O_{\imath} = 1$ equal to $P_{defeat}(\imath)$, the probability that unit $\imath$ is defeated, $\imath = 1, \ldots, m$. Given m units, there are 2^m possible outcomes.

Generally, the commander will desire at least a certain percentage of units be defeated in order for the employment of nuclear weapons to be considered effective. We can define another binary function of the random variables $\underline{O}$, $\phi(\underline{O})$, such that $\phi(\underline{O}) = 1$ if the commander's objective is met; 0 otherwise. Clearly $\phi(\underline{O})$ is nondecreasing in $\underline{O}$. The function ϕ may be regarded as identical to a structure function of a coherent system in reliability theory (Barlow and Proschan [1981]); thus we can use results from coherent structure theory in our analysis of the nuclear exchange issue.

For example, if any k out of m units must be defeated in order for the commander's objective to be met, $\phi(\underline{O}) = (O_1 O_2 \cdots O_k) \coprod (O_1 O_2 \cdots O_{k-1} O_{k+1}) \coprod \cdots \coprod (O_{m-k+1} \cdots O_m)$,

for all possible subsets of size k from the m units, $1 \leq k < m$,

where $(x_{\jmath}) \coprod (x_{\imath}) = 1 - (1 - x_{\imath})(1 - x_{\jmath})$.

Furthermore, we can bound $P[\,\phi(\underline{O})=1\,]$ by (Barlow & Proschan [1981] p. 31):

$$\max_{1\,\le\,r\,\le\,npath}\ \prod_{i\in P_r} P[\,O_i=1\,] \ \le\ P[\,\phi(\underline{O})=1\,] \ \le\ \min_{1\,\le\,s\,\le\,ncut}\ \coprod_{i\in K_s} P[\,O_i=1\,]\ ,$$

where P_r denotes one of the $npath =\begin{pmatrix} m \\ k \end{pmatrix}$ possible min path sets (in this case, a min path set is any set of k units), K_s denotes one of the $ncut =\begin{pmatrix} m \\ m\text{-}k\text{+}1 \end{pmatrix}$ possible min cut sets (in this case, a min cut set is any set of m-k+1 units), and $\coprod_i X_i = 1 - \prod_i (1-X_i)$. If we let $p_o(i) = P[\,O_i=1\,]$, and number the units such that $p_o(1) \le p_o(2) \le \cdots \le p_o(m)$, then

$$\max_{1\,\le\,r\,\le\,npath}\ \prod_{i\in P_r} P[O_i=1] = \prod_{i=m\text{-}k\text{+}1}^{m} p_o(i)\ ; \quad \min_{1\,\le\,s\,\le\,ncut}\ \coprod_{i\in K_s} P[O_i=1] = \coprod_{i=1}^{m\text{-}k\text{+}1} p_o(i)\ .$$

This example shows how we can estimate (through bounds) the probability that the commander's objective may be met. It may be the case, however, that it makes a difference in the battle that follows the nuclear exchange *which* units are defeated in the exchange. Or, more simply, it may be *how many* units are defeated across the theater which makes a difference.

For example, suppose that there are 20 opposing divisions in a sector of combat. Our best judgement, given the tactical and operational situation, is that the defeat of at least 7 divisions out of the 20 will be required to avoid loss of territory (stabilize the FLOT - which may be the commander's objective). However, if 14 or more divisions are defeated, an opportunity occur, not merely to stabilize the FLOT but also to conduct a successful counterattack. In this case, if $O_i = 1$ of division i is defeated, $i = 1, \ldots, 20$, there are 2^{20} possible outcomes. We can partition the sample space of possible outcomes into the $\sum_{k=0}^{6} \begin{pmatrix} 20 \\ k \end{pmatrix}$ outcomes where 6 or fewer divisions are defeated, the $\sum_{k=7}^{13} \begin{pmatrix} 20 \\ k \end{pmatrix}$ outcomes where 7 or more but less than 14 divisions are defeated, and the $\sum_{k=14}^{20} \begin{pmatrix} 20 \\ k \end{pmatrix}$ outcomes where 14 or more divisions are defeated. From each of the three partitions so created, one realization can be selected to be used as input to a theater-level combat simulation such as FORCEM. FORCEM or a similar model will then be run three times using each of the three nuclear outcomes as an input. If our assumption about the impact of defeating different numbers of divisions is correct, the three battles simulated in FORCEM using different outcomes should yield noticeably different results. The response surface estimated using these three FORCEM runs should provide a better representation of the variability possible in theater-level combat where nuclear weapons are employed than a random selection of three possible outcomes from the 2^{20} possible (and certainly better than selecting a single FORCEM run).

Estimating the sample space of all possible outcomes as part of the input to a low resolution deterministic theater-level simulation is discussed in more detail in Youngren [1989c].

CHAPTER 4
SUMMARY

In this paper, we have provided a model for representing tactical nuclear warfare at the theater level. It provides either an analytic or a Monte Carlo solution to the representation of the effect of the exchange on each potential nuclear target. In doing so, the model addresses two current problems in modeling nuclear weapons exchanges at the theater level.

The first problem is the identification of the locations of small, lower level units, such as companies or battalions, within theater-level conventional scenarios or models which track units at the divisional level. When target aggregation and preclusion are considered, the actual location attributed to these units will make a difference in determining what weapons may be used to engage which units in the theater.

The current approaches to identifying these small unit locations are to either specify stylized arrays or templates, based on doctrine, which are then applied to all divisional-sized units, or to manually generate a single, specific array. Both of these approaches fail to take into account the variability inherent in the actual locations that might be realized in any specific battle.

Our solution to this problem is to treat the small unit locations as unknown, and to describe our uncertainty about these locations through probability distributions. We start with prior multinormal distributions for small unit locations based on expert opinion, and then update that information by generating many different array realizations. We carry forward these distributions to account for the possible shifts in Desired Ground Zero (DGZ) due to target location errors, aggregation, and preclusion. Errors in weapon delivery systems are accounted for through the distribution defined by the Circular Error Probable (CEP) and are matched with the DGZ distributions to form distributions for the Actual Ground Zeros (AGZ) for nuclear weapons that may be employed within the theater. From the distributions of the AGZ and the unit location, the distribution for the level of damage achieved against the unit can be derived.

The second major problem lies in the interface between theater-level nuclear analyses, which may use the probabilistic arrays developed above, and theater conventional battle simulations,

which tend to be expected value models. An expected value model demands a single input to represent the effect of a nuclear exchange, which is carried forward into simulation of the post-nuclear battle. However, a theater-level nuclear exchange may generate many different outcomes which will have a significantly different effect upon any post-nuclear battle. We use the probability models developed in this paper to estimate sets of possible nuclear exchange outcomes, which can be partitioned into sets that have a significantly different effect on the conventional battle. The expected value simulation can be run several times, once for each set of outcomes, to capture the variability inherent in the nuclear exchange and predict the effects of that variability upon the conventional battle.

The probability models described in this paper may be used in three different ways. First, they may be used as a research tool to estimate the sensitivity of exchange outcomes to the various data and assumptions included in the model. Second, they may be used as a surrogate for detailed, complex simulation models of nuclear exchanges such as NUFAM III. Finally, the models may be used to estimate the sample space of all possible outcomes of a theater nuclear exchange to decide which outcomes should be provided to theater-level expected value models.

Implementation

In order to implement the model, we begin with the steps given immediately below. These are followed by procedures which vary, depending upon how the model is to be used.

1. Based on a conventional theater-level scenario, or the output from a conventional theater-level model, determine the appropriate time to model the use of theater nuclear weapons. This is a judgment that becomes part of a study or analysis framework.

2. From the conventional theater-level scenario or model output, extract information on major unit (e.g., divisions) locations and strengths. Determine the appropriate parameters to use to model the acquisition and movement within the timeframe of the nuclear exchange. These may be generated from a separate model, using the scenario at the time of the exchange (for additional information, see Youngren [1989 b,c]).

3. Match the appropriate multinormal distributions to each of the major units in order to describe the probability distributions of the locations of the subordinate units (e.g., companies and

battalions). The distributions will be based upon a doctrinal template for that unit, which specifies the mean locations for the units, and variance/covariance matrices generated from expert opinion and data gathered from manual arrays.

a. Use as a Research Tool

If we are going to use the model as a research tool to estimate various parameters pertaining to specific units and see how these parameters vary with different input data, assumptions, etc., we would continue to implement the model as follows:

1) For the units of interest, use the procedures contained in this paper to calculate measures of interest such as $p_{engage}(i, s \mid w)$, $p_{engage}(i, \overline{s} \mid w)$, $p_{engage}(ij, s \mid w)$, and $p_{engage}(ij, \overline{s} \mid w)$. This can either be done directly, using the procedures outlined in Chapter 3, or through a Monte Carlo estimation technique (Appendix E). The input data and assumptions can be varied as desired to determine the impact on these measures.

2) For the units of interest, determine the probability that unit i can be defeated, $p_{defeat}(i)$. This can either be done directly or through a Monte Carlo estimation technique. The input data and assumptions can be varied as desired to determine the impact on the defeat probability.

b. Use as a Surrogate for Detailed Simulation Models

If we are going to use the model as a surrogate for detailed simulation models to estimate the effect of a nuclear exchange, we can implement the model as a stochastic simulation:

1) Use the multinormal probability distributions for unit locations to generate realizations of actual and perceived unit locations.

2) For each unit, draw against the probability p_{avail} that it is acquired and can be retained as a target at least until the time of detonation. The procedures for calculating these probabilities may be found in Youngren [1989a,b].

3) For the acquired units, determine if they can be engaged by various weapons w, taking into account aggregation and preclusion issues.

4) Allocate the available weapons against the realized set of available targets, using any preferred allocation scheme.

5) Draw against the weapon reliability and delivery errors to determine the AGZs for the weapon. Assess damages resulting from detonations occurring at those points (a further draw may be made against the probability of a "dud" round, if desired). For each unit, determine if it was defeated.

6) The results of the damage assessment represent the effects of the exchange against targeted units, and may be analyzed accordingly. As a stochastic simulation, this process should be repeated multiple times with different random number streams.

Alternatively, we can use the model as a surrogate for detailed simulation models using the probability estimates:

7) For each unit, determine the probability p_{avail} that it is acquired and can be retained as a target at least until the time of detonation.

8) Use the procedures contained in this paper to calculate measures of interest such as $p_{engage}(i, s \mid w)$, $p_{engage}(i, \overline{s} \mid w)$, $p_{engage}(ij, s \mid w)$, and $p_{engage}(ij, \overline{s} \mid w)$. For accuracy, we recommend that a Monte Carlo estimation technique be used. This estimation procedure need only be performed once, provided that the distributions of the unit locations do not change. Use the p_{avail} probability to help determine the target unit pairs to be considered for aggregation.

9) Using the probabilities that the units are available for engagement, determine the probability that unit i can be defeated, $p_{defeat}(i)$. This can either be done directly or through a Monte Carlo estimation technique.

10) As an alternative to step #9 above, we can generate a realization of an engagement list by drawing against $p_{engage}(i \mid w)$, $p_{engage}(ij \mid w)$, and $p_{avail}(i)$. We can then allocate weapons against this realized set and assess damages. As we are dealing with a single realization at this point, this part of the analysis should be replicated.

c. Use to Generate Inputs to a Theater Level Expected Value Model

If we are going to use the model to generate inputs representing the effects of a nuclear exchange into a theater-level expected value model, we would implement the model as follows:

1) For all units, use the procedures contained in this paper to calculate measures of interest such as $p_{engage}(i, s \mid w)$, $p_{engage}(i, \overline{s} \mid w)$, $p_{engage}(ij, s \mid w)$, and $p_{engage}(ij, \overline{s} \mid w)$. This can either be done directly or through a Monte Carlo estimation technique.

2) For all units, determine the probability that unit i can be defeated, $p_{defeat}(i)$. This can either be done directly or through a Monte Carlo estimation technique.

3) Aggregate the probabilities that each target unit can be defeated into the probabilities that each larger unit represented in the theater-level model (e.g., divisions) can be defeated, using defeat criteria established in the study or analysis. This establishes the space of all possible outcomes of the nuclear exchange (in terms of the binary defeat events).

4) Partition the space of all possible outcomes of the nuclear exchange into sets (strata) that we expect to lead to significantly different outcomes at the theater level.

5) Select the modal outcome from each strata to determine which units should be acquired, retained and be available for fire planning.

6) For each strata, generate a realization of an engagement set, such that the units available for eng ment are defined by the modal outcome from the strata.

7) Allocate the available weapons against the engagement set of targets, using any preferred allocation scheme (Appendix F).

8) Draw against the weapon delivery errors to determine the AGZs for the weapon. Assess damages resulting from detonations occurring at those points. For each unit, the AGZ is constrained to the defeat outcome determined in the selection from the strata.

9) The results of the damage assessment represent the effects of the exchange against targeted units that will be provided to the theater-level expected value model. This realization is used to represent the effects of all possible outcomes within the strata from which it was selected.

10) Repeat steps 6 through 9 for each modal outcome from each strata.

Conclusions

The procedures outlined in this paper may be used in a variety of ways to analyze the effects of possible theater nuclear engagements. Three advantages are realized from using these procedures: First, the uncertainty surrounding the precise locations of the targeted units is explicitly accounted for through multivariate probability distributions. Second, the proposed procedures form a model that is significantly less complex than the detailed simulations currently used to perform theater nuclear analysis, and these procedures may be applied incrementally to single units without having to run an entire simulation. Third, it is possible to construct an experimental design to estimate the variability in FORCEM outputs.

These procedures will be implemented at CAA in a model called NEMESIS, which may be used as a stand-alone model or may be used to prepare input to the agency's theater-level combat model. FORCEM.

APPENDIX A
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APPENDIX B

PROBABILITY DEFINITIONS

Probabilities related to target acquisition and movement

$p_{avail}(i)$

Let $p_{avail}(i)$ be the probability that a target unit i is acquired and retained as a target until detonation, and is stationary if required.

$$p_{avail} = P[\, Y_r > T_d - t_p \mid \text{acquired at time } t_p\,] \cdot P[\text{ acquired at } t_p\,] \cdot \{\, p_{stay}\,\}^I$$

where $I = 0$ or 1 depending upon the requirement for the unit to be stationary at the time of acquisition.

a. No Capability Exists to Observe the Target after Acquisition

$$p_{avail} = P[\, Y_r > T_d - t_p \mid \text{on acquisition list at } t_p \text{ and stationary at A }]$$
$$\cdot P[\text{ unit on the acquisition list at } t_p\,] \cdot P[\text{ unit stationary at A }]$$

where A is the time the target was acquired

b. The Target is Observed Periodically after Acquisition

$$p_{avail} = P[\, Y_r > T_d - t_p \mid \text{on acquisition list at } t_p \text{ and stationary at } T_l\,]$$
$$\cdot P[\text{ unit on the acquisition list at } t_p\,] \cdot P[\text{ unit stationary at } T_l\,]$$

where T_l is the last time the target was observed prior to time t_p

c. The Target is Observed Continuously after Acquisition with Preplanned Fire

$$p_{avail} = P[\, Y_r > T_d - t_p \mid \text{on acquisition list at } t_p\,] \cdot P[\text{ unit on the acquisition list at } t_p\,]$$

d. The Target is Observed Continuously after Acquisition without Preplanned Fire

$$p_{avail} = 1 \cdot p_{acq}$$

p_{hit}

We have the *opportunity* to hit a unit if it is available for fire and it is stationary at time T_d at the place where it was last observed.

a. No Capability Exists to Observe the Target after Acquisition

$$p_{hit} = P[\ Y_r > T_d - t_p \mid \text{on acquisition list at } t_p\ ] \cdot P[\ \text{unit on the acquisition list at } t_p\ ]$$
$$\cdot P[\ Y_S > T_d - A \mid \text{stationary at A}\ ] \cdot P[\ \text{unit stationary at A}\ ].$$

b. The Target is Observed Periodically after Acquisition

$$p_{hit} = P[\ Y_r > T_d - t_p \text{ and } Y_S > T_d - T_l \mid \text{on acquisition list at } t_p \text{ and stationary at } T_l\ ]$$
$$\cdot P[\ \text{unit on the acquisition list at } t_p\ ] \cdot P[\ \text{unit stationary at } T_l\ ],$$

c. The Target is Observed Continuously after Acquisition with Preplanned Fire

$$p_{hit} = P[\ Y_r > T_d - t_p \mid \text{on acquisition list at } t_p\ ] \cdot P[\ \text{unit on the acquisition list at } t_p\ ]$$
$$\cdot P[\ Y_S > T_d - t_p \mid \text{stationary at } t_p\ ] \cdot P[\ \text{unit stationary at } t_p\ ].$$

d. The Target is Observed Continuously after Acquisition without Preplanned Fire

$$p_{hit} = P[\ Y_r > T_d - T_f \mid \text{on acquisition list at } T_f\ ] \cdot P[\ \text{unit on the acquisition list at } T_f\ ]$$
$$\cdot \left\{ P[\ Y_s > \eta \mid \text{stationary at } t_p\ ] \cdot p_{stay} \right.$$
$$\left. + P[\ S > \eta \mid \text{stopped at } T_f\ ] \cdot P[\ Y_m < L - \eta \mid \text{moving at } t_p\ ] \cdot p_{move} \right\}$$

Probabilities related to location

$\overline{p}_{prec}(\ i \mid w,\ a_i,\ \overline{agg}_i\ )$

Let $\overline{p}_{prec}(\ i \mid w, a_i,\ \overline{agg}_i\ )$ equal the probability that the unit will *not* be precluded from engagement, given that it is available, can be engaged using weapon w, and it is not suitable for engagement as an aggregate target; i.e., $D_{ij}^2 > (\ d_{i\,w} + d_{j\,w}\ )^2 = d_{ij\,w}^2$ or j was not available $\forall\ j \neq i$. However, when we approximate joint probabilities, we ignore the conditioning when evaluating $P[\ \underset{k}{\cap}\ D_{iP_k}^2 > r_{WP_k}^2\ ]$. Thus

$\overline{p}_{prec}(i \mid w,\ a_i, \overline{agg}_i\ ) \equiv P[\ \text{unit } i \text{ will } not \text{ be precluded} \mid \text{available, weapon } w,\ D_{ij}^2 > d_{ij\,w}^2\ ]$.

$\overline{p}_{prec}(i \mid w,\ a_i, \overline{agg}_i\ ) \doteq P[\ \underset{k}{\cap}\ D_{iP_k}^2 > r_{WP_k U_i}^2\ ]$, bounded from above as:

$\overline{p}_{prec}(i \mid w,\ a_i, \overline{agg}_i\ ) \doteq \min\left\{\ P[\ D_{iP_k}^2 > r_{WP_k U_i}^2\ ]\ \right\}_{k=1}^{np}.$

$\overline{p}_{aggr}(\ w \mid \text{available}\ )$

Let $G_{ij\,w} = 1$ if $D_{ij}^2 \leq (d_{i\,w} + d_{j\,w})^2 = d_{ij\,w}^2$, given weapon w and both units $i,\ j$ available.
$\qquad = 0$ otherwise.

$\overline{p}_{aggr}(\ w \mid \text{available}\ ) \equiv P(\ \text{no aggregation with weapon } w \mid \text{all units available})$
$$\doteq \min\left\{\ P[\ G_{ij\,w} = 0\ ]\ \right\}_{i,j=1}^{m}$$

$\overline{p}_{aggr}(\,i\,|\,w,\,a_i\,)$

Let $\overline{p}_{aggr}(\,i\,|\,w,\,a_i\,)$ be the probability that *no* unit can be aggregated with unit i, given that weapon w is used and unit i is available. This can occur when $D_{ij}^2 > d_{ijw}^2$ or when j is not available for fire planning.

$\overline{p}_{aggr}(\,i\,|\,w,\,a_i\,) \equiv P(\text{ no aggregation involving unit } i \,|\, \text{weapon } w;\, i \text{ available })$

$$= P[\,\{\,\underset{j}{\cup}\,\{\,(\,D_{ij} \le d_{ijw}\,) \cap (\,j \text{ available }) \cap (\,\underset{k}{\cap} D_{jP_k} \ge {}^r W_{P_k}\,U_j\,)\,\}\,\}^c \,|\, w,\, a_i\,]$$

$$= P[\,\underset{j}{\cap}\,\{\,(\,D_{ij} \le d_{ijw}\,) \cap (\,j \text{ available }) \cap (\,\underset{k}{\cap} D_{jP_k} \ge {}^r W_{P_k}\,U_j\,)\,\}^c \,|\, w,\, a_i\,]$$

$$\doteq \min\Big\{\,P[\,\{\,(\,D_{ij} \le d_{ijw}\,) \cap (\,j \text{ available }) \cap (\,\underset{k}{\cap} D_{jP_k} \ge {}^r W_{P_k}\,U_j\,)\,\}^c \,|\, w,\, a_i\,]\,\Big\}_{\substack{j=1 \\ j \ne i}}^m$$

$$\doteq \min\Big\{\,1 - P[\,(\,D_{ij} \le d_{ijw}\,) \cap (\,j \text{ available }) \cap (\,\underset{k}{\cap} D_{jP_k} \ge {}^r W_{P_k}\,U_j\,) \,|\, w,\, a_i\,]\,\Big\}_{\substack{j=1 \\ j \ne i}}^m$$

$$\doteq 1 - \max\Big\{\,P[D_{ij} \le d_{ijw}\,|\,w,\,a_{ij}] \cdot P[\,j \text{ available }] \cdot P[\,\underset{k}{\cap} D_{jP_k} \ge {}^r W_{P_k}\,U_j\,|\,w,\,a_{ij}]\,\Big\}_{\substack{j=1 \\ j \ne i}}^m$$

$$\doteq 1 - \max\Big\{\,p_{aggr}(\,ij\,|\,w,\,a_{ij}) \cdot P_{avail}(j) \cdot \overline{p}_{prec}(j\,|\,w,\,a_j,\,\overline{agg}_j\,)\,\Big\}_{\substack{j=1 \\ j \ne i}}^m$$

$p_{aggr}(\,ij\,|\,w,\,a_{ij}\,)$

Let $p_{aggr}(\,ij\,|\,w,\,a_{ij})$ be the probability that units i and j can be aggregated, given that weapon w is used and units i and j are available.

$$p_{aggr}(\,ij\,|\,w,\,a_{ij}) \equiv P[\,D_{ij}^2 \le (\,d_{iw} + d_{jw}\,)^2 \,|\, w;\, i,\, j \text{ available }] = P[\,G_{ijw} = 1\,]$$

$\overline{p}_{aggr}(\,ij\,|\,w,\,a_{ij}\,)$

Let $\overline{p}_{aggr}(\,ij\,|\,w,\,a_{ij})$ be the probability that units i and j *cannot* be aggregated, given that weapon w is used and units i and j are available.

$$\overline{p}_{aggr}(\,ij\,|\,w,\,a_{ij}) \equiv P[\,D_{ij}^2 > (\,d_{iw} + d_{jw}\,)^2 \,|\, w;\, i \text{ and } j \text{ available }]$$

$$= P[\,G_{ijw} = 0\,] = 1 - p_{aggr}(\,ij\,|\,w,\,a_{ij}\,)$$

$P_{no\ shift}(\,i\,|\,w,\,a_i,\,\overline{agg}_i\,)$

Let $P_{no\ shift}(\,i\,|\,w,\,a_i,\,\overline{agg}_i\,)$ be the probability that there is no shift in DGZ (due to preclusion) for target i, given that weapon w is used, unit i is available, and there are no suitable aggregate targets. By definition, if there is no DGZ shift due to preclusion, the unit i is not precluded. As in $\overline{p}_{prec}(i\,|\,w,\,a_i,\overline{agg}_i\,)$, we condition on $D_{ij}^2 > (\,d_{iw} + d_{jw}\,)^2$ but ignore the conditioning when evaluating the joint probabilities. Thus $P_{no\ shift}(\,i\,|\,w,\,a_i,\,\overline{agg}_i\,) \equiv P[\text{ no shift in DGZ for target } i \,|\, \text{weapon } w,\, i \text{ available, no aggregation }] = P[\,\underset{k}{\cap} D_{iP_k}^2 > {}^r{}_{WP_k}^2\,]$

$$P_{no\ shift}(\,i\,|\,w,\,a_i,\,\overline{agg}_i\,) \doteq \min\Big\{\,P[\,D_{iP_k}^2 > {}^r{}_{WP_k}^2\,]\,\Big\}_{k=1}^{np}.$$

$p_{shift}(i \,|\, w, a_i, a\overline{g}g_i)$

Let $p_{shift}(i \,|\, w, a_i, a\overline{g}g_i)$ be the probability that target unit i can be engaged as a single target with a shift in DGZ due to preclusion, given that weapon w is used and unit i is available. Again, we condition on no suitable aggregate targets, $D_{ij}^2 > (d_{iw} + d_{jw})^2$ but ignore the conditioning when evaluating the joint probabilities.

$$p_{shift}(i \,|\, w, a_i, a\overline{g}g_i) = P\left(\left\{ \bigcap_k D_{iP_k}^2 > r_{WP_kU_i}^2 \right\} \cap \left\{ \bigcap_k D_{iP_k}^2 > r_{WP_k}^2 \right\}^C \right)$$

$$= P[\bigcap_k D_{iP_k}^2 > r_{WP_kU_i}^2] - P[\bigcap_k D_{iP_k}^2 > r_{WP_k}^2]$$

$$p_{shift}(i \,|\, w, a_i, a\overline{g}g_i) \doteq \min\left\{ P[D_{iP_k}^2 > r_{WP_kU_i}^2] \right\}_{k=1}^{np} - \min\left\{ P[D_{iP_k}^2 > r_{WP_k}^2] \right\}_{k=1}^{np}$$

Note that $p_{shift}(i \,|\, w, a_i, a\overline{g}g_i) + p_{no\ shift}(i \,|\, w, a_i, a\overline{g}g_i)$

$$= P[\bigcap_k D_{iP_k}^2 > r_{WP_kU_i}^2] = \overline{p}_{prec}(i \,|\, w, a_i, a\overline{g}g_i)$$

$\overline{p}_{prec}(ij \,|\, w, a_{ij}, agg_{ij})$

Let $\overline{p}_{prec}(ij \,|\, w, a_{ij}, agg_{ij})$ equal the probability that *both* target units i and j are *not* precluded from engagement, given that both are available, both can be aggregated with each other, and they can be jointly engaged using weapon w. In this case, we condition on $D_{ij}^2 \le (d_{iw} + d_{jw})^2 = d_{ijw}^2$:

$$\overline{p}_{prec}(ij \,|\, w, a_{ij}, agg_{ij}) \equiv P[\bigcap_k \left\{ D_{DP_k}^2 > r_{WP_kU_i}^2 \cap D_{DP_k}^2 > r_{WP_kU_j}^2 \right\} \,|\, D_{ij}^2 \le d_{ijw}^2],$$

where $D_{DP_k}^2 = $ the distance from the unshifted aggregate target (X_{DGZ}, Y_{DGZ}) to preclusion area k; that is, $D_{DP_k}^2 = (X_{DGZ} - X_{P_k})^2 + (Y_{DGZ} - Y_{P_k})^2$. We approximate as

$$\overline{p}_{prec}(ij \,|\, w, a_{ij}, agg_{ij}) \doteq \min\left\{ \left\{ P[D_{DP_k}^2 > r_{WP_kU_l}^2 \,|\, D_{ij}^2 \le d_{ijw}^2] \right\}_{l=i}^{j} \right\}_{k=1}^{np}, \text{ where}$$

$$P[D_{DP_k}^2 > r_{WP_kU_l}^2 \,|\, D_{ij}^2 \le d_{ijw}^2] = 1 - \frac{1 - P[D_{DP_k}^2 > r_{WP_kU_l}^2]}{P[D_{ij}^2 \le d_{ijw}^2]} \text{ for any } i,j \text{ and for } l=i \text{ or } j.$$

$p_{no\ shift}(ij \,|\, w, a_{ij}, agg_{ij})$

Let $p_{no\ shift}(ij \,|\, w, a_{ij}, agg_{ij})$ be the probability that there is no shift in DGZ (due to preclusion) for the aggregate target formed from units i and j, given that weapon w is used, both units i and j are available, and they can be aggregated with each other.

$$p_{no\ shift}(ij \,|\, w, a_{ij}, agg_{ij}) \equiv P[\bigcap_k D_{DP_k}^2 > r_{WP_k}^2 \,|\, D_{ij}^2 \le d_{ijw}^2]$$
$$\doteq \min\left\{ P[D_{DP_k}^2 > r_{WP_k}^2 \,|\, D_{ij}^2 \le d_{ijw}^2] \right\}_{k=1}^{np}.$$

$P_{shift}(ij \mid w, a_{ij}, agg_{ij})$

Let $p_{shift}(ij \mid w)$ be the probability that there *is* a shift in DGZ due to preclusion for the aggregate target formed from units i and j and the shifted DGZ is still within the maximum offset distances d_{iw} and d_{jw}, given that weapon w is used, both units i and j are available, and they can be aggregated with each other.

$$P_{shift}(ij \mid w, a_{ij}, agg_{ij}) =$$

$$P\left(\left\{ \bigcap_k D^2_{DP_k} > r^2_{WP_k U_{ij}} \right\} \cap \left\{ \bigcap_k D^2_{DP_k} > r^2_{WP_k} \right\}^C \cap \left\{ D^2_{iD^s} \le d^2_{iw} \right\} \cap \left\{ D^2_{jD^s} \le d^2_{jw} \right\} \mid D^2_{ij} \le d^2_{ij\,w} \right) .$$

$$P_{shift}(ij \mid w, a_{ij}, agg_{ij}) = \left[P[\bigcap_k D^2_{iP_k} > r^2_{WP_k U_{ij}} \mid D^2_{ij} \le d^2_{ij\,w}] - P[\bigcap_k D^2_{iP_k} > r^2_{WP_k} \mid D^2_{ij} \le d^2_{ij\,w}] \right]$$

$$\cdot P[D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw} \mid D^2_{ij} \le d^2_{ij\,w}],$$

where $P[D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw} \mid D^2_{ij} \le d^2_{ij\,w}] = \dfrac{1 \cdot P[D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw}]}{P[D^2_{ij} \le d^2_{ij\,w}]}.$

Note that $p_{shift}(ij \mid w, a_{ij}, agg_{ij}) + P_{no\ shift}(ij \mid w, a_{ij}, agg_{ij}) \le P[\bigcap_k D^2_{DP_k} > r^2_{WP_k U_{ij}}] = \overline{p}_{prec}(ij \mid w, a_{ij})$ due to the correction factor $P[D^2_{iD^s} \le d^2_{iw} \cap D^2_{jD^s} \le d^2_{jw} \mid D^2_{ij} \le d^2_{ij\,w}].$

Probabilities related to weapon allocation

$P_{round}(w \mid a_{ij})$

Let $p_{round}(w \mid a_{ij})$ be the probability that weapon type w can be allocated against unit i, given that unit i is available.

$P_{round}(w \mid a_{ij})$

Let $p_{round}(w \mid a_{ij})$ be the probability that weapon type w can be allocated against the aggregate target formed from units i and j, given that the aggregate target formed from i and j is available.

Probabilities of engagement

$P_{engage}(i, s \mid w)$

Let $p_{engage}(i, s \mid w)$ be the probability that unit i is available for engagement using weapon w with a DGZ shift, given that weapon w is available to engage the target.

$$P_{engage}(i, s \mid w) = P[\ i \text{ engaged as a single target with DGZ shift} \mid w\ ]$$

$$= P[\ i \text{ available}\ ] \cdot P[\{ \bigcap_k D_{iP_k} \ge r_{WP_k U_i} \} \cap \{ \bigcap_k D_{iP_k} \ge r_{WP_k} \}^C \mid w\ ]$$

$$\cdot P[\{ \bigcup_j \{ (D_{ij} \le d_{ijw}) \cap (j \text{ available}) \cap (\bigcap_k D_{jP_k} \ge r_{WP_k U_j}) \} \}^C \mid w\ ]$$

$$P_{engage}(i, s \mid w) = p_{avail}(i) \cdot p_{shift}(i \mid w, a_i, \overline{agg}_i) \cdot \overline{p}_{aggr}(i \mid w, a_i)$$

$p_{engage}(i, s, w)$

Let $p_{engage}(i, s, w)$ be the probability that unit i is available for engagement using weapon w with a DGZ shift.

$$p_{engage}(i, s, w) = p_{avail}(i) \cdot p_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) \cdot p_{round}(w \mid a_i)$$

$p_{engage}(i, \overline{s} \mid w)$

Let $p_{engage}(i, \overline{s} \mid w)$ be the probability that unit i is available for engagement using weapon w without a DGZ shift, given that weapon w is available to engage the target.

$$p_{engage}(i, \overline{s} \mid w) = P[\,i \text{ engaged as a single target with no DGZ shift} \mid w\,]$$
$$= P[\,i \text{ available}\,] \cdot P[\,\underset{k}{\cap} D_{iP_k} \geq {}^r WP_k \mid w\,]$$
$$\cdot P[\,\left\{ \underset{j}{\cup} \left\{ (D_{ij} \leq d_{ijw}) \cap (j \text{ available}) \cap (\underset{k}{\cap} D_{jP_k} \geq {}^r WP_k \, U_j) \right\} \right\}^c \mid w\,]$$
$$p_{engage}(i, \overline{s} \mid w) = p_{avail}(i) \cdot p_{no\ shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i)$$

$p_{engage}(i, \overline{s}, w)$

Let $p_{engage}(i, \overline{s}, w)$ be the probability that unit i is available for engagement using weapon w without a DGZ shift.

$$p_{engage}(i, \overline{s}, w) = p_{avail}(i) \cdot p_{no\ shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) \cdot p_{round}(w \mid a_i)$$

$p_{engage}(ij, s \mid w)$

Let $p_{engage}(ij, s \mid w)$ be the probability that the aggregate target formed from units i and j is engaged as an aggregate target with a DGZ shift, given that weapon w is used.

$$p_{engage}(ij, s \mid w) = P[\,i \text{ engaged as an aggregate target with DGZ shift} \mid w\,]$$
$$= P[\,i \text{ available}\,] \cdot P[\,j \text{ available}\,]$$
$$\cdot P[\,\left\{ \underset{k}{\cap} D_{DP_k} \geq {}^r WP_k \, U_{ij} \right\} \cap \left\{ \underset{k}{\cap} D_{DP_k} \geq {}^r WP_k \right\}^c \mid D_{ij} \leq d_{ijw} \mid w, a_{ij}\,]$$
$$\cdot P[\,\left\{ D^2_{iD^s} \leq d^2_{iw} \right\} \cap \left\{ D^2_{jD^s} \leq d^2_{jw} \right\} \mid D_{ij} \leq d_{ijw}, w, a_{ij}\,] \cdot P[\,D_{ij} \leq d_{ijw} \mid w, a_{ij}\,]$$
$$p_{engage}(ij, s \mid w) = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij})$$

$p_{engage}(ij, s, w)$

Let $p_{engage}(ij, s, w)$ be the probability that the aggregate target formed from units i and j is engaged as an aggregate target with a DGZ shift using weapon w.

$$p_{engage}(ij, s, w) = p_{avail}(i) \cdot p_{avail}(j)$$
$$\cdot p_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij}) \cdot p_{round}(w \mid a_{ij})$$

$p_{engage}(\,ij, \overline{s}\mid w\,)$

Let $p_{engage}(\,ij, \overline{s}\mid w\,)$ be the probability that the aggregate target formed from units i and j is engaged as an aggregate target with no DGZ shift, given that weapon w is used.

$p_{engage}(\,ij, \overline{s}\mid w\,) = P[\ i$ engaged as an aggregate target with no DGZ shift $\mid w\ ]$

$= P[\,i\ \text{available}] \cdot P[\,j\ \text{available}] \cdot P[\{\ \underset{k}{\cap}\ D_{DP_k} \geq r_{WP_k}\ \cap\ \{\ D_{ij} \leq d_{ijw}\ \}\mid w,\, a_{ij}\ ]$

$= P[\,i\ \text{available}] \cdot P[\,j\ \text{available}] \cdot P[\ \underset{k}{\cap}\ D_{DP_k} \geq r_{WP_k} \mid D_{ij} \leq d_{ijw},\, w,\, a_{ij}\ ] \cdot P[\ D_{ij} \leq d_{ijw}\mid w,\, a_{ij}\ ]$

$p_{engage}(\,ij, \overline{s}\mid w\,) = p_{avail}(\,i\,) \cdot p_{avail}(\,j\,) \cdot p_{no\ shift}(\,ij\mid w,\, a_{ij},\, agg_{ij}\,) \cdot p_{aggr}(\,ij\mid w,\, a_{ij}\,)$

$p_{engage}(\,ij, \overline{s}, w\,)$

Let $p_{engage}(\,ij, \overline{s}, w\,)$ be the probability that the aggregate target formed from units i and j is engaged as an aggregate target with a DGZ shift using weapon w.

$p_{engage}(\,ij, \overline{s}, w\,) = p_{avail}(\,i\,) \cdot p_{avail}(\,j\,) \cdot p_{round}(\,w\mid a_{ij}\,) \cdot p_{no\ shift}(\,ij\mid w,\, a_{ij},\, agg_{ij}\,)$

$\cdot\ p_{aggr}(\,ij\mid w,\, a_{ij}\,)$

Probabilities of conditional defeat

$p_{defeat}(\,i\mid w,\, a_i,\, \overline{s},,\, a\overline{g}g_i\,)$

Let $p_{defeat}(\,i\mid w,\, a_i,\, \overline{s},\, a\overline{g}g_i\,) \equiv P[\ \text{unit}\ i\ \text{defeated}\mid\ \text{weapon}\ w;\ \text{no DGZ shift};\ i\ \text{available};\ \text{no}$ aggregation] be the probability that unit i can be defeated as a single target, given that weapon w is used, there is no DGZ shift, and unit i is available for fire planning. If unit i is engaged as a single target with no DGZ shift, then the DGZ is located at the perceived target center, with coordinates $X_{DGZ} = X_{iL}$ and $Y_{DGZ} = Y_{iL}$. Using weapon w, unit i is defeated if $(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2 \leq d_{iw}^2$. Thus

$p_{defeat}(\,i\mid w,\, a_i,\, \overline{s},\, a\overline{g}g_i\,) = P[\ (\,X_{AGZ} - X_i\,)^2 + (\,Y_{AGZ} - Y_i\,)^2 \leq d_{iw}^2\ ]$

$p_{defeat}(\,i\mid w,\, a_i,\, s,\, a\overline{g}g_i\,)$

Let $p_{defeat}(\,i\mid w,\, a_i,\, s,\, a\overline{g}g_i\,) \equiv P[\ \text{unit}\ i\ \text{defeated}\mid\ \text{weapon}\ w;\ \text{DGZ shift};\ i\ \text{available};\ \text{no}$ aggregation] be the probability that unit i can be defeated as a single target, given that weapon w is used, there is a DGZ shift, and unit i is available for fire planning. If unit i is engaged as a single target with a shifted DGZ, then the DGZ is located at the shifted coordinates $X_{DGZ}^s = \beta X_{iL} + (1 - \beta) X_{P_k}$ and $Y_{DGZ}^s = \beta Y_{iL} + (1 - \beta) Y_{P_k}$. Using weapon w, unit i is defeated if $(X_{AGZ}^s - X_i)^2 + (Y_{AGZ}^s - Y_i)^2 \leq d_{iw}^2$. Thus

$p_{defeat}(\,i\mid w,\, a_i,\, s,\, a\overline{g}g_i\,) = P[\ (\,X_{AGZ}^s - X_i\,)^2 + (\,Y_{AGZ}^s - Y_i\,)^2 \leq d_{iw}^2\ ].$

$p_{defeat}(\, ij \mid w, a_i, \overline{s}, agg_{ij} \,)$

Let $p_{defeat}(\, ij \mid w, a_i, \overline{s}, a\overline{g}g_i \,) \equiv P[$ units i and j defeated as aggregate target $\mid$ weapon w; no DGZ shift; i,j available $]$ be the probability that units i and j can be defeated as an aggregate target, given that weapon w is used, there is no DGZ shift, and units i and j are available for fire planning. If units i and j are engaged as an aggregate target with no DGZ shift, then the DGZ is located along a line segment connecting the perceived target centers, with coordinates $X_{DGZ} = \alpha\, X_{iL} + (1-\alpha)\, X_{jL}$ and $Y_{DGZ} = \alpha\, Y_{iL} + (1-\alpha)\, Y_{jL}$. Using weapon w, unit i is defeated if $(\, X_{AGZ} - X_i \,)^2 + (\, Y_{AGZ} - Y_i \,)^2 \leq d_{iw}^2$ and unit j is defeated if $(\, X_{AGZ} - X_j \,)^2 + (\, Y_{AGZ} - Y_j \,)^2 \leq d_{jw}^2$. Thus

$$p_{defeat}(\, ij \mid w, a_i, \overline{s}, a\overline{g}g_i \,)$$
$$= P[\, \{ (X_{AGZ}-X_i)^2 + (Y_{AGZ}-Y_i)^2 \leq d_{iw}^2 \} \cap \{ (X_{AGZ}-X_j)^2 + (Y_{AGZ}-Y_j)^2 \leq d_{jw}^2 \} \,]$$

$p_{defeat}(\, ij \mid w, a_i, s, agg_{ij} \,)$

Let $p_{defeat}(\, ij \mid w, a_i, s, a\overline{g}g_i \,) \equiv P[$ units i and j defeated as aggregate target $\mid$ weapon w; DGZ shift; i,j available$]$ be the probability that units i and j can be defeated as an aggregate target, given that weapon w is used, there is a DGZ shift, and units i and j are available for fire planning. If units i and j are engaged as an aggregate target with a DGZ shift due to preclusion, then the DGZ is shifted from the point located along a line segment connecting the perceived target centers, with shifted coordinates $X_{DGZ}^s = \beta\, X_{DGZ} + (1-\beta)\, X_{P_k}$ and $Y_{DGZ}^s = \beta\, Y_{DGZ} + (1-\beta)\, Y_{P_k}$, where the unshifted DGZ had coordinates $X_{DGZ} = \alpha\, X_{iL} + (1-\alpha)\, X_{jL}$ and $Y_{DGZ} = \alpha\, Y_{iL} + (1-\alpha)\, Y_{jL}$. Using weapon w, unit i is defeated if $(X_{AGZ}^s - X_i)^2 + (Y_{AGZ}^s - Y_i)^2 \leq d_{iw}^2$ and unit j is defeated if $(\, X_{AGZ}^s - X_j \,)^2 + (\, Y_{AGZ}^s - Y_j \,)^2 \leq d_{jw}^2$. Thus

$$p_{defeat}(\, ij \mid w, a_i, s, a\overline{g}g_i \,)$$
$$= P[\, \{ (X_{AGZ}^s-X_i)^2 + (Y_{AGZ}^s-Y_i)^2 \leq d_{iw}^2 \} \cap \{ (X_{AGZ}^s-X_j)^2 + (Y_{AGZ}^s-Y_j)^2 \leq d_{jw}^2 \} \,]$$

$p_{defeat}(\, i \mid w, a_i, \overline{s}, agg_{ij} \,)$

Let $p_{defeat}(\, i \mid w, a_i, \overline{s}, a\overline{g}g_i \,) \equiv P[$ unit i defeated as aggregate target $\mid$ weapon w; no DGZ shift; i,j available $]$ be the probability that unit i can be defeated as an aggregate target, given that weapon w is used, there is no DGZ shift, and units i and j are available for fire planning. If units i and j are engaged as an aggregate target with no DGZ shift, then the DGZ is located along a line segment connecting the perceived target centers, with coordinates X_{DGZ} and Y_{DGZ} as given previously. Using weapon w, unit i is defeated if $(X_{AGZ} - X_i)^2 + (Y_{AGZ} - Y_i)^2 \leq d_{iw}^2$. Thus

$$p_{defeat}(\, i \mid w, a_i, \overline{s}, a\overline{g}g_i \,) = P[(X_{AGZ}-X_i)^2 + (Y_{AGZ}-Y_i)^2 \leq d_{iw}^2]$$

$p_{defeat}(i \mid w, a_i, s, agg_{ij})$

Let $p_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) \equiv$ P[unit i defeated as aggregate target | weapon w; DGZ shift; i,j available] be the probability that unit i can be defeated as an aggregate target, given that weapon w is used, there is a DGZ shift, and units i and j are available for fire planning. If units i and j are engaged as an aggregate target with a DGZ shift due to preclusion, then the DGZ is shifted from the point located along a line segment connecting the perceived target centers, with shifted coordinates $X_{DGZ}^s = \beta\, X_{DGZ} + (1-\beta)\, X_{P_k}$ and $Y_{DGZ}^s = \beta\, Y_{DGZ} + (1-\beta)\, Y_{P_k}$, where the unshifted DGZ had coordinates X_{DGZ} and Y_{DGZ} as given previously. Using weapon w, unit i is defeated if $(X_{AGZ}^s - X_i)^2 + (Y_{AGZ}^s - Y_i)^2 \le d_{iw}^2$. Thus

$$p_{defeat}(ij \mid w, a_i, s, a\overline{g}g_i) = P[(X_{AGZ}^s - X_i)^2 + (Y_{AGZ}^s - Y_i)^2 \le d_{iw}^2$$

Probabilities of defeat

$p_{defeat}(i \mid w)$

Let $p_{defeat}(i \mid w) \equiv$ P[unit i defeated | weapon w] be the probability that unit i is available and can be defeated, given that weapon w is used. Thus we are averaging over the probability that unit i is available, that it is engaged as a single or aggregate target, and that the DGZ is or is not shifted.

$p_{defeat}(i \mid w)$

$= $ P[unit i is defeated as a single target $\mid w$] $+$ P[unit i is defeated as an aggregate target $\mid w$]

$= $ P[$D_{Ai} \le d_{iw} \mid$ DGZ shift, i single target, w] $\cdot$ P[DGZ shift, i single target $\mid w$]

$+$ P[$D_{Ai} \le d_{iw} \mid$ no DGZ shift, i single target, w] $\cdot$ P[no DGZ shift, i single target $\mid w$]

$+$ P[$D_{Ai} \le d_{iw} \mid$ DGZ shift, ij aggregate target, w] $\cdot$ P[DGZ shift, ij aggregate target $\mid w$]

$+$ P[$D_{Ai} \le d_{iw} \mid$ no DGZ shift, ij aggregate target, w] $\cdot$ P[no DGZ shift, ij aggregate target $\mid w$]

$p_{defeat}(i \mid w)$

$= p_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) \cdot p_{avail}(i) \cdot p_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i)$

$+ p_{defeat}(i \mid w, a_i, \overline{s}, a\overline{g}g_i) \cdot p_{avail}(i) \cdot p_{no\,shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i)$

$+ p_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) \cdot p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij})$

$+ p_{defeat}(i \mid w, a_{ij}, \overline{s}, agg_{ij}) \cdot p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\,shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij})$

$p_{defeat}(i \mid w)$

$= p_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) \cdot p_{engage}(i, s \mid w) + p_{defeat}(i \mid w, a_i, \overline{s}, a\overline{g}g_i) \cdot p_{engage}(i, \overline{s} \mid w)$

$+ p_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) \cdot p_{engage}(ij, s \mid w) + p_{defeat}(i \mid w, a_{ij}, \overline{s}, agg_{ij}) \cdot p_{engage}(ij, \overline{s} \mid w)$

$P_{defeat}(i, w)$

Let $P_{defeat}(i, w) \equiv P[$ unit i defeated using weapon w] be the probability that unit i is available and can be defeated using weapon w. In this case, we remove the conditioning on weapon w.

$P_{defeat}(i, w)$

$$= \Big\{ P_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) \cdot P_{avail}(i) \cdot P_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{P}_{aggr}(i \mid w, a_i)$$

$$+ P_{defeat}(i \mid w, a_i, \overline{s}, a\overline{g}g_i) \cdot P_{avail}(i) \cdot P_{no\ shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{P}_{aggr}(i \mid w, a_i) \Big\}$$

$$\cdot P_{round}(w \mid a_i)$$

$$+ P_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) \cdot P_{avail}(i) \cdot P_{avail}(j) \cdot P_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot P_{aggr}(ij \mid w, a_{ij})$$

$$+ P_{defeat}(i \mid w, a_{ij}, \overline{s}, agg_{ij}) \cdot P_{avail}(i) \cdot P_{avail}(j) \cdot P_{no\ shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot P_{aggr}(ij \mid w, a_{ij}) \Big\}$$

$$\cdot P_{round}(w \mid a_{ij})$$

$P_{defeat}(i, w)$

$$= P_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) \cdot P_{engage}(i, s, w) + P_{defeat}(i \mid w, a_i, \overline{s}, a\overline{g}g_i) \cdot P_{engage}(i, \overline{s}, w)$$

$$+ P_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) \cdot P_{engage}(ij, s, w) + P_{defeat}(i \mid w, a_{ij}, \overline{s}, agg_{ij}) \cdot P_{engage}(ij, \overline{s}, w)$$

$P_{defeat}(i)$

Let $P_{defeat}(i)$ be the probability that unit i is defeated using tactical nuclear weapons.

$$P_{defeat}(i) = \sum_w P[\text{ Unit } i \text{ defeated using weapon } w]$$

$$P_{defeat}(i) = \sum_w P_{defeat}(i, w)$$

APPENDIX C

MULTINORMAL DISTRIBUTIONS

Section I. UPDATING THE MULTIVARIATE DISTRIBUTIONS FOR UNIT LOCATIONS*

The Multivariate Normal Distribution

We use the following notation to specify the m-dimensional *multinormal probability density function*, or pdf:

$$ f(x \mid \mu, \Sigma) = \left[\frac{1}{2\pi} \right]^{-\frac{m}{2}} |\Sigma|^{-\frac{m}{2}} \exp\left\{ -\frac{1}{2} (x - \mu)^t \Sigma^{-1}(x - \mu) \right\}, \qquad \infty < x < \infty. $$

where $\mu \equiv (\mu_1, \mu_2, ..., \mu_m)^t$ is an m-dimensional known mean vector and Σ is a ($m \times m$) - dimensional positive definite symmetric variance-covariance matrix. We define a matrix $R \equiv \Sigma^{-1}$. R is referred to as the *precision matrix* of the distribution. μ is assumed known but R is assumed unknown with a multivariate Wishart prior.

Multivariate Wishart Distribution

If $S_1, ..., S_n$ are a random sample of m-dimensional random vectors with mean vector 0 and and $m \times m$ variance/covariance matrix Σ_w, and V is defined such that

$$ V = \sum_{i=1}^{n} S_i S_i^t, $$

then the random matrix V has a *Wishart distribution with n degrees of freedom and parametric matrix Σ_w*, where $n > m - 1$ and Σ_w is nonsingular. For any $m \times m$ matrix v which is symmetric and positive definite, the pdf of v is:

$$ f(v \mid n, \Sigma_w) = c |\Sigma|^{-\frac{n}{2}} |v|^{\frac{(n-m-1)}{2}} \exp\left\{ -\frac{1}{2} \text{tr}(\Sigma^{-1} v) \right\}, $$

where $\text{tr}(\Sigma^{-1} v)$ denotes the trace of the matrix $\Sigma^{-1} v$ and the constant c is equal to:

$$ c = \left[2^{\frac{nk}{2}} \pi^{\frac{k(k-1)}{4}} \prod_{j=1}^{m} \Gamma\left(\frac{n+1-j}{2} \right) \right]^{-1}. $$

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* Material in this section has been extracted from DeGroot [1970]

Obtaining the Posterior Distribution

DeGroot[1970, pp. 176,177] provides the following result: Suppose that X_1, ..., X_n are a random sample of m-dimensional random vectors with a specified mean vector m and and an unknown value of the $m \times m$ precision matrix R. Suppose also that the prior distribution of R is a Wishart distribution with α degrees of freedom and precision matrix τ such that $\alpha > m-1$ and τ is a symmetric positive definite matrix. Then the posterior distribution of R when $X_i = x_i$ ($i = 1$, ..., n) is a Wishart distribution with $\alpha + n$ degrees of freedom and precision matrix τ^*, where

$$\tau^* = \tau + \sum_{i=1}^{n} (x_i - m)(x_i - m)^t.$$

We can use this result to easily update the posterior distribution of the unknown precision matrix R of our multinormal distribution for the X and Y coordinates of the actual unit locations, given data X_1, ..., X_n and Y_1, ..., Y_n that represent n vectors of manual array locations.

Obtaining the Prior Distribution Without Data

If the initial (prior) distribution cannot be specified by an expert in arraying, then a noninformative prior distribution can be used. In essence, a noninformative prior for the precision matrix R is an improper Wishart distribution with $\alpha = 0$ degrees of freedom and precision matrix $\tau = 0$. Therefore, given a random sample of data vectors x_1, ..., x_n, the posterior distribution of R is a Wishart distribution with $n-1$ degrees of freedom and precision matrix τ^*, where

$$\tau^* = \sum_{i=1}^{n} (x_i - m)(x_i - m)^t.$$

If R is distributed as Wishart with n degrees of freedom and precision matrix τ (thus variance/covariance matrix τ^{-1}), then the expectation of R, $E[R] = n\tau^{-1}$. For our multinormal distributions used in our probability model for the actual unit locations, we use the multinormal pdf given the known vector of unit mean locations μ_i and the estimated variance/covariance $\Sigma = [n\tau^{-1}]^{-1} = \frac{\tau}{n}$:

$$f(x \mid \mu) = \left[\frac{1}{2\pi} \right]^{-\frac{m}{2}} \left|\frac{\tau}{n}\right|^{\frac{1}{2}} \exp\{ -\frac{1}{2} (x - \mu)^t \frac{\tau}{n} (x - \mu) \}, \qquad \infty < x < \infty.$$

Obtaining the Prior Distribution With Data

If the initial (prior) distribution can be specified by an expert in arraying, then a proper Wishart distribution will be used. We assume the variables X to be distributed as multinormal (μ, Σ). where μ is known (the templated mean locations) and $\Sigma = E[\ R^{-1}\ ]$. From the Wishart distribution, we know that $E[\ R\ ] = (\alpha+n)\ \tau^{-1}$. Thus $\Sigma = E[\ R^{-1}\ ] = E[\ \{(\alpha+n)\ \tau^{*-1}\}^{-1}\ ]$ $= \frac{1}{\alpha+n}\ \tau$. The updating formula given on the previous page will be used to obtain the posterior Wishart distribution for R.

Section II. GENERATING MULTINORMAL PSEUDORANDOM VARIATES*

Procedure

Let $X^t = (\ X_1, X_2, ..., X_m\ )$ be an m-dimensional vector of random variables with a joint pdf

$$f(\ x\mid\mu, \Sigma) = \left[\ \frac{1}{2\pi}\ \right]^{-\frac{m}{2}} |\Sigma|^{-\frac{m}{2}} \exp\{\ -\frac{1}{2}(\ x - \mu\ )^t \Sigma^{-1}(\ x - \mu\ )\ \}, \qquad \infty < x < \infty.$$

Then X can be represented as

$$X = C\ Z + \mu,$$

where C is a unique lower triangular matrix solving $\Sigma = C\ C^t$.

Let c_{ij} and σ_{ij} denote elements in the ith row and jth column of C and Σ respectively. Then Algorithm LTM from Fishman is used to compute C.

Algorithm LTM

1. $a = \sqrt{\sigma_{11}}$

2. For $i = 1, ..., m$, $c_{i1} = \frac{\sigma_{i1}}{a}$

3. $i = 2$

4. $c_{ii} = \left[\ \sigma_{ii} - \sum_{j=1}^{i-1} c_{ij}^2\ \right]^{\frac{1}{2}}$

5. If $i = m$, stop.

6. Otherwise, $i = i + 1$

7. For $j = 2, ..., i-1$, $c_{ij} = \dfrac{\sigma_{ij} - \sum_{l=1}^{j-1} c_{il}\ c_{jl}}{c_{jj}}$

8. Go to 4.

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* Material in this section has been extracted from Fishman [1978]

Generating Multinormal Pseudorandom Variates

To generate m-dimensional vectors $\mathbf{x}$ from a multinormal distribution, given μ and Σ, algorithm MN1 from Fishman is used.

Algorithm MN1

1. For $i = 1, \ldots, m$, generate N(0,1) pseudorandom variates Z_i.

2. $i = 1$

3. $X_i = \mu_i + \sum_{j=1}^{i} c_{ij} Z_j$

4. If $i = m$, stop

5. Otherwise, $i = i + 1$

6. Go to 3

APPENDIX D

DISTRIBUTIONS FOR QUADRATIC FORMS IN NORMAL VARIABLES

Johnson and Kotz [1972] discuss approximations to quadratic forms in normal variables. If we define $F_n(\,q;\,\underline{\lambda},\,\underline{\omega}\,) \equiv P[\,Q(\,\underline{Z}\,) \leq q\,]$ for given n, then we can establish approximations for $F_2(\,q;\,\underline{\lambda},\,\underline{0}\,)$ (the central case) and $F_2(\,q;\,\underline{\lambda},\,\underline{\omega}\,)$ (the general case).

Notation

$$Q(\,\underline{Z}\,) = \sum_{j=1}^{n} \lambda_j\,(\,Z_j - \omega_j)^2, \text{ where } Z_j \sim N(\,0,1\,).$$

$$F_n(\,q;\,\underline{\lambda},\,\underline{\omega}\,) = P[\,Q(\,\underline{Z}\,) \leq q\,]$$

$$D_{ij}^2 = D_{xij}^2 + D_{yij}^2 = \sigma_{xij}^2\,(\,Z_{xij} + \frac{\mu_{xij}}{\sigma_{xij}}\,)^2 + \sigma_{yij}^2\,(\,Z_{yij} + \frac{\mu_{yij}}{\sigma_{yij}}\,)^2$$

$$Z_{kij} \sim N(\,0,1\,) \text{ for } k = x,\,y.$$

$$\lambda_k = \sigma_{kij}^2 \text{ for } k = x,\,y.$$

$$\omega_k = \frac{\mu_{kij}}{\sigma_{kij}} \text{ for } k = x,\,y.$$

For the perceived unit location $(\,X_{iL},\,Y_{iL}\,)$,

$$\mu_{xij} \equiv (\,\mu_{xi} + \mu_{txi} - \mu_{xj} - \mu_{txj}\,)$$

$$\sigma_{xij}^2 \equiv (\,\sigma_{xi}^2 + \sigma_{txi}^2 + \sigma_{xj}^2 + \sigma_{txj}^2 - 2\rho_{xij}\sigma_{xi}\sigma_{xj}\,).$$

$$\mu_{yij} \equiv (\,\mu_{yi} + \mu_{tyi} - \mu_{yj} - \mu_{tyj}\,)$$

$$\sigma_{yij}^2 \equiv (\,\sigma_{yi}^2 + \sigma_{tyi}^2 + \sigma_{yj}^2 + \sigma_{tyj}^2 - 2\rho_{yij}\sigma_{yi}\sigma_{yj}\,).$$

The Central Case

If we consider the special case where all $\omega = 0$, and we order our λ's such that $\lambda_1 \geq \lambda_2$, then we can find $F_2(\,q;\,\underline{\lambda},\,\underline{0}\,)$ using a result by Rubin [1960]:

$$F_2(\,q;\,\underline{\lambda},\,\underline{0}\,) = P[\,\chi_2'^2(\,r^2\,) \leq R^2\,] - P[\,\chi_2'^2(\,R^2\,) \leq r^2\,]\,, \text{ where}$$

$\chi_2'^2(\,r^2\,)$ denotes a non-central chi-square random variable with 2 degrees of freedom and non-centrality parameter r^2,

$$R = \frac{1}{2}\left[\left(\frac{q}{\lambda_1}\right)^2 + \left(\frac{q}{\lambda_2}\right)^2\right], \quad r = \frac{1}{2}\left[\left(\frac{q}{\lambda_1}\right)^2 - \left(\frac{q}{\lambda_2}\right)^2\right], \text{ and}$$

$$P[\chi_2^{'2}(r^2) \leq R^2] \doteq \frac{R^2}{2 + \frac{R^2}{2}} \exp\left[\frac{-r^2}{2 + \frac{R^2}{2}}\right]$$

The Non-Central Case

If we consider the general case where not all $\omega = 0$, we use an approach was developed by Rubin [1962]:

$$F_n(q; \underline{\lambda}, \underline{\mu}_z) = \sum_{j=0}^{\infty} e_j' \, P[\chi_{n+2j}^2 \leq \frac{q}{\beta}], \text{ where}$$

β = any arbitrary constant; we suggest $\beta = \frac{2\lambda_x\lambda_y}{\lambda_x + \lambda_y}$,

$$e_0' = \left[\exp\left(-\frac{1}{2}\sum_{k=x}^{y}\omega_k\right)\prod_{k=x}^{y}\left[\frac{\beta}{\lambda_k}\right]^{\frac{1}{2}}\right],$$

$$e_r' = \frac{1}{2r}\sum_{j=0}^{r-1}G_{r-j}\, e_j' \quad, \text{ and}$$

$$G_r = \sum_{k=x}^{y}\left[1 - \frac{\beta}{\lambda_k}\right]^r + r\beta\sum_{k=x}^{y}\left[\frac{\omega_k^2}{\lambda_k}\right]\left[1 - \frac{\beta}{\lambda_k}\right]^{r-1} \quad (r \geq 1).$$

For β, λ_k and ω_k defined as given above,

$$e_0' = \frac{\sigma_{xij}\sigma_{yij}}{\sigma_{xij}^2 + \sigma_{yij}^2}\exp\left\{-\frac{1}{2}\left(\frac{\mu_{xij}}{\sigma_{xij}} + \frac{\mu_{xij}}{\sigma_{xij}}\right)\right.$$

$$e_r' = \frac{1}{2r}\sum_{j=0}^{r-1}G_{r-j}\, e_j' \quad, \text{ and for } r \geq 1),$$

$$G_r = \left[1 - \frac{2\sigma_{kij}^2}{\sigma_{xij}^2 + \sigma_{yij}^2}\right]r + \left[1 - \frac{2\sigma_{kij}^2}{\sigma_{xij}^2 + \sigma_{yij}^2}\right]r$$

$$+ \frac{2r\sigma_{xij}^2\sigma_{yij}^2}{\sigma_{xij}^2 + \sigma_{yij}^2}\cdot\left[\frac{\mu_{xij}}{\sigma_{xij}^2}\right]2\cdot\left[1 - \frac{2\sigma_{yij}^2}{\sigma_{xij}^2 + \sigma_{yij}^2}\right]r-1$$

$$+ \frac{2r\sigma_{xij}^2\sigma_{yij}^2}{\sigma_{xij}^2 + \sigma_{yij}^2}\cdot\left[\frac{\mu_{yij}}{\sigma_{yij}^2}\right]2\cdot\left[1 - \frac{2\sigma_{xij}^2}{\sigma_{xij}^2 + \sigma_{yij}^2}\right]r-1.$$

To evaluate the central chi-square cdf, we recommend the Wilson-Hilferty [1931] approximation:

$$P[\ \chi^2_{n+2j} \leq \tfrac{q}{\beta}\ ] \doteq \Phi\left[\ \left\{ \left(\frac{\frac{q}{\beta}}{n+2j}\right)^3 - 1 + \tfrac{2}{9}\ (\ n+2j)^{-1}\right\}\ \sqrt{\frac{9\,(n+2j)}{2}}\ \right] \qquad \begin{array}{l} j=0,1,\ldots \\ n=0,1,\ldots \end{array}$$

To evaluate, we determine as many of the terms of the sum as necessary; perhaps the first three terms. The error of approximation is also found in Rubin [1962].

A Less Accurate but Simpler Approximation

A simpler approximation can be derived if we are willing to accept less accuracy. It is known that the limiting distribution of a standardized chi-square is $N[\ 0,1\ ]$. If we approximate the chi-square variable with its limiting distribution, we can obtain the following for any non-central chi-square $\chi^2_{\nu}(\omega)$ (the central chi-square is a special case), where $Z \sim N(0,1)$ (Johnson & Kotz [1972, p 141]):

$$F(\ x\ ;\ \nu,\ \omega\ ) = P[\ \chi^2_{\nu}(\omega) \leq x\ ] = P\left[\ Z \leq \frac{x - \nu - \omega}{[\ 2(\ \nu + 2\omega\ )\ ]^{1/2}}\ \right]$$

$$P[\ \chi^2_{\nu}(\omega) \leq x\ ] = P[\ Z\ [\ 2(\ \nu + 2\omega\ )\ ]^{1/2} + \nu + \omega\ \leq x\ ]$$

Recall that $D^2_{kij} = \sigma^2_{kij}\ \chi^2_{\nu}(\omega^2_k)$, where $\omega^2_k = \dfrac{\mu^2_{kij}}{\sigma^2_{kij}}$. Thus

$$D^2_{kij} \doteq \sigma^2_{kij}\left[\ Z\ [\ 2(\ \nu + 2\omega^2_k\ )\ ]^{1/2} + \nu + \omega^2_k\ \right]$$

or $D^2_{kij} \doteq \left[\ \sigma^2_{kij}\ [\ 2(\ \nu + 2\omega^2_k\ )\ ]^{1/2}\ \right]Z + (\nu + \omega^2_k)\sigma^2_{kij}$,

which is of the form $aZ + b$, a Normal distributed variable. Recalling that $\nu = 1$ and $\omega^2_k = \dfrac{\mu^2_{kij}}{\sigma^2_{kij}}$,

$$D^2_{kij} \sim N\left[\ \sigma^2_{kij} + \mu^2_{kij},\ 2\sigma^2_{kij}(\sigma^2_{kij} + 2\mu^2_{kij})\ \right]\ \text{for } k = x,y.$$

$D^2_{ij} = D^2_{xij} + D^2_{yij}$; therefore

$$D^2_{ij} \sim N\left[\ \sigma^2_{xij} + \mu^2_{xij} + \sigma^2_{yij} + \mu^2_{yij},\ 2\sigma^2_{xij}(\sigma^2_{xij} + 2\mu^2_{xij}) + 2\sigma^2_{yij}(\sigma^2_{yij} + 2\mu^2_{yij})\ \right].$$

It is easy to verify that the approximation is unbiased with respect to the first and second moments.

To evaluate,

$$P[\ D_{ij}^2 \leq d\ ] = P\left[\ Z \leq \frac{d - (\sigma_{xij}^2 + \mu_{xij}^2 + \sigma_{yij}^2 + \mu_{yij}^2)}{\sqrt{2\sigma_{xij}^2(\sigma_{xij}^2 + 2\mu_{xij}^2) + 2\sigma_{yij}^2(\sigma_{yij}^2 + 2\mu_{yij}^2)}}\ \right].$$

Evaluating Bivariate Normal CDFs

If X_1, $X_2 \sim BVN(\ (0,0),\ (1,1),\ \rho\ )$ [standardized bivariate normal], then we define $L(\ h,k,\rho\ )$ as $L(\ h,k,\rho\ ) \equiv P[\ X_1 > h;\ X_2 > k\ ]$.

It is easy to verify that

$$F_{X_1X_2}(h,k) = P[\ X_1 \leq h;\ X_2 \leq k\ ] = P[\ X_1 > h\ ] + P[\ X_2 > k\ ] + P[\ X_1 > h;\ X_2 > k\ ] - 1.$$

Pearson [1901] (see also Johnson & Kotz [1972], p. 118) suggested the following approximation to $L(\ h,k,\rho\ )$:

$$L(\ h,k,\rho\ ) \doteq \Phi(h)\ \Phi(k) + Z(h)\ Z(k)\left[\ \rho + \frac{\rho^2}{2!}hk + \frac{\rho^3}{3!}(h-1)(k-1) + \cdots\ \right],$$

where $Z(x) = \frac{1}{\sqrt{2\pi}}\ e^{-\frac{x^2}{2}}$ and $\Phi(x) = \int_0^x Z(u)\ du$.

Let Y_1, $Y_2 \sim BVN(\ (\mu_1,\mu_2),\ (\sigma_1^2,\ \sigma_2^2),\ \rho\ )$ and let X_1, X_2 be defined as $X_1 \equiv \frac{Y_1 - \mu_1}{\sigma_1}$. $X_2 \equiv \frac{V_2 - \mu_2}{\sigma_2}$. Then

$$F_{Y_1Y_2}(\ y_1,y_2) = P[\ Y_1 \leq y_1;\ Y_2 \leq y_2\ ] = P[\ \frac{Y_1 - \mu_1}{\sigma_1} \leq \frac{y_1 - \mu_1}{\sigma_1};\ \frac{Y_2 - \mu_2}{\sigma_2} \leq \frac{y_2 - \mu_2}{\sigma_2}\ ]$$

$$= P[\ X_1 \leq \frac{y_1 - \mu_1}{\sigma_1};\ X_2 \leq \frac{y_2 - \mu_2}{\sigma_2}\ ]$$

$$= \Phi[\ \frac{y_1 - \mu_1}{\sigma_1}\ ] + \Phi[\ \frac{y_2 - \mu_2}{\sigma_2}\ ] + L[\ \frac{y_1 - \mu_1}{\sigma_1},\ \frac{y_2 - \mu_2}{\sigma_2}\ ] - 1$$

From Pearson's approximation, using the first two terms,

$$F_{Y_1Y_2}(\ y_1,y_2) \doteq \Phi(h) + \Phi(k) + \Phi(h)\ \Phi(k) + Z(h)\ Z(k)\left[\ \rho + \frac{\rho^2}{2!}hk\ \right] - 1,$$

where $h = \frac{y_1 - \mu_1}{\sigma_1}$, $k = \frac{y_2 - \mu_2}{\sigma_2}$.

APPENDIX E

MONTE CARLO EXPERIMENTS

1. Generate an array of actual unit locations (X_i, Y_i) from the multinormal distribution.

2. Read in the preclusion area locations within the larger unit area and convert terrain coordinates to local coordinates X_{P_k}, Y_{P_k}.

3. For each unit $i = m,...,1$ by -1

4.   Generate TLE random variables TX_i, TY_i. Add to (X_i, Y_i) to get perceived unit locations X_{iL}, Y_{iL}.

5.   For each preclusion area $k = 1,...,np$

6.   Compute the distance from perceived unit i to preclusion area k, D_{iP_k}.

7.   For each weapon type $w = 1,...,nw$

       [Determine if unit i is precluded from engagement with weapon w]

8.   If $D_{iP_k} \leq r_{WP_k U_i}$

9.   Set a binary variable $B_{WP_k U}(i,w) = 1$

10.   Exit to step 13 (next weapon w).

       [If not precluded, determine if DGZ shifts due to preclusion. Find closest preclusion area]

11.   Otherwise, if $D_{iP_k} < r_{WP_k}$ and $D_{iP_k} < D_{minP_k}$,

12.   Set a variable $D_{minP_k}(i) = D_{iP_k}$ and an integer variable $I_{WP_k}(i) = k$

13.   Loop on weapon w.

14.   Loop on preclusion area k.

15. Loop on unit i.

16. For each unit $i = m,...,1$ by -1

17.   For each weapon type $w = 1,...,nw$

18.   If $B_{WP_k U}(i,w) = 0$, [Unit i not precluded]

19.   For each unit $j = i+1,...,m$,

20.   If $B_{WP_k U}(j,w) = 0$, [Unit j not precluded]

21.   Compute the distance from unit i to unit j, $D_{ij} = D_{ji}$.

           [Determine if i and j can be aggregated. Find closest unit j to i]

22.   If $D_{ij} \leq (d_{iw} + d_{jw})$ [i,j can be aggregated]

23.   If $D_{ij} < D_{min\,aggr}(i,w)$,

24.   Set a variable $D_{min\,aggr}(i,w) = D_{ij}$

25.   Set an integer variable $I_{aggr}(i,w) = j$

26.   If $D_{ij} < D_{min\,aggr}(j,w)$

27.   Set a variable $D_{min\,aggr}(j,w) = D_{ij}$

28.   Set an integer variable $I_{aggr}(j,w) = i$

29.   Loop on unit j.

30.   Loop on weapon w.

31. Loop on unit i.

32. For each unit $i = 1,\ldots,m$

33.   For each weapon type $w = 1,\ldots,nw$

34.   If $B_{WP_kU}(i,w) = 1$ [[preclusion]],

35.   Exit to step 88 (next weapon w).

36.   If $I_{aggr}(i,w) > m$ [[already aggregated with a previously considered unit i]]

37.   Exit to step 88 (next weapon w).

38.   If $I_{aggr}(i,w) = 0$,

39.   Exit to step 74 below (single target).

        [[Compute aggregated DGZ]]

40.   Otherwise, [[$I_{aggr}(i,w) \neq 0$, $j = I_{aggr}(i,w)$, and $D_{ij} = D_{min\,aggr}(i,w)$]]

41.   Calculate $\alpha = \dfrac{D_{ij}-d_{iw}}{D_{ij}}$ and compute $X_{DGZ}(i,w) = X_{DGZ}(j,w)$

                 $= \alpha\, X_{iL} + (1-\alpha)\, X_{jL}$, $Y_{DGZ}(i,w) = Y_{DGZ}(j) = \alpha\, Y_{iL} + (1-\alpha)\, Y_{jL}$.

42.   For each preclusion area k,

43.   Compute the distance from aggregate target ij (at coordinates $X_{DGZ}(i,w)$,

            $Y_{DGZ}(i,w)$) to preclusion area k, D_{DP_k}.

            [[Check to see if aggregate DGZ is precluded]]

44.   If $D_{DP_k} \leq r_{WP_kU_{ij}}$,

45.   Set $I_{aggr}(i,w) = 0$ [[preclusion rules out aggregate DGZ for weapon w]]

46.   Exit to step 74 below (single target).

47.   Otherwise [[$D_{DP_k} > r_{WP_kU_{ij}}$]]

            [[Check to see if aggregate DGZ must be shifted due to preclusion]]

48.   If $D_{DP_k} < r_{WP_k}$ and $D_{DP_k} < T_{D\,min\,P_k}$,

49.   Set a temporary variable $T_{D\,min\,P_k} = D_{DP_k}$

50.   Set an integer variable $I_{D\,WP_k}(i,w) = k$

51.   Loop on preclusion area k.

52.   If $I_{D\,WP_k}(i,w) = 0$, [No shift in DGZ due to preclusion]

53.   Set a binary variable $B_{no\,shift\,aggr}(i,w) = B_{no\,shift\,aggr}(j,w) = 1$

54.   Set $I_{aggr}(j,w) = m + i$.

55.   Exit to step 88   (next weapon w).

56.   Otherwise [$I_{D\,WP_k}(i,w) \neq 0$, aggregate target, no preclusion, DGZ shift,

       where $j = I_{aggr}(i,w)$, $k = I_{D\,WP_k}(i,w)$, and $D_{DP_k} = T_{D\,min\,P_k}$]

57.   Calculate $\beta = \dfrac{r_{WP_k}}{D_{DP_k}}$ and compute new $X_{DGZ}(i,w) = X_{DGZ}(j,w) =$

       $\beta\,X_{DGZ}(j,w) + (1-\beta)\,X_{P_k}$, new $Y_{DGZ}(i,w) = Y_{DGZ}(j,w) =$

       $\beta\,Y_{DGZ}(j,w) + (1-\beta)\,Y_{P_k}$.

       [Check to see if shifted DGZ still covers both targets]

58.   Compute the distance from shifted aggregate target ij DGZ (at coordinates

       $X_{DGZ}(i,w)$, $Y_{DGZ}(i,w)$) to target i, $D_{i\,D^s}$ and target j, $D_{j\,D^s}$.

59.   If $D_{i\,D^s} > d_{iw}$ or $D_{j\,D^s} > d_{jw}$

60.   Set $I_{aggr}(i,w) = 0$ (shift required by preclusion rules out aggregate DGZ)

61.   Reset $X_{DGZ}(i,w)$, $X_{DGZ}(j,w)$ to X_{iL}, X_{jL}

       and $Y_{DGZ}(i,w)$, $Y_{DGZ}(j,w)$ to Y_{iL}, Y_{jL}

62.   Exit to step 74 below (single target).

63.   Else for all preclusion areas k,

       [Check to see if shifted DGZ is ruled out due to preclusion]

64.   Compute the distance from shifted aggregate target ij DGZ (at coordinates

       $X_{DGZ}(i,w)$, $Y_{DGZ}(i,w)$) to preclusion area k, $D_{D^s\,P_k}$.

65.   If $D_{D^s\,P_k} \leq r_{WP_k}U_{ij}$,

66.   Set $I_{aggr}(i,w) = 0$ (preclusion rules out aggregate DGZ)

67.   Reset $X_{DGZ}(i,w)$, $X_{DGZ}(j,w)$ to X_{iL}, X_{jL}

       and $Y_{DGZ}(i,w)$, $Y_{DGZ}(j,w)$ to Y_{iL}, Y_{jL}

68.   Exit to step 74 below (single target).

69.   Loop on preclusion area k.

70.   Set a binary variable $B_{shift\,aggr}(i,w) = B_{shift\,aggr}(j,w) = 1$.

71.   Set $I_{aggr}(j,w) = m + i$.

72.   End IF [aggregate target, no preclusion, DGZ shift]

73.   Exit to step 88 (next weapon w)

74.   [Single target, no preclusion, $I_{aggr}(i,w) = 0$, $B_{WP_kU}(i,w) = 0$]

75.   Set a binary variable $B_{no\,aggr}(i,w) = 1$ [Unit i not aggregated]

76.   If $I_{WP_k}(i,w) = 0$,

77.   Set a binary variable $B_{no\ shift}(i,w) = 1$

78.   Exit to step 88 (next weapon w)

79.   Else $[\![\ I_{WP_k}(i,w) \neq 0$, single target, no preclusion, DGZ shift, where $k = I_{DWP_k}(i)\ ]\!]$

80.   Calculate $\beta = \dfrac{r_{WP_k}}{D_{min\,P_k}(i)}$ and compute $X_{DGZ}(i,w) = \beta\,X_{iL} + (1-\beta)\,X_{P_k}$,

                $Y_{DGZ}(i,w) = \beta\,Y_{iL} + (1-\beta)\,Y_{P_k}$.

81.   For all preclusion areas k,

82.   Compute the distance from shifted target i DGZ (at coordinates $X_{DGZ}(i,w)$,

                $Y_{DGZ}(i,w)$) to preclusion area k, $D_{i^\prime P_k}$.

                $[\![$ Check to see if shifted DGZ violates any other preclusion area restrictions $]\!]$

83.   If $D_{i^\prime P_k} \leq r_{WP_k U_i}$,

84.   Reset $X_{DGZ}(i,w)$, $X_{DGZ}(j,w)$ to X_{iL}, X_{jL}

                      and $Y_{DGZ}(i,w)$, $Y_{DGZ}(j,w)$ to Y_{iL}, Y_{jL}

85.   Exit to step 88 (next weapon w)

86.   Otherwise, set a binary variable $B_{shift}(i,w) = 1$.

87.   Loop on preclusion area k.

If desired, the probability of defeating each unit i given weapon w can be estimated here

    Generate CEP random variables CX_i, CY_i. Add to $X_{DGZ}(i,w)$ and $Y_{DGZ}(i,w)$ respectively to get $X_{AGZ}(i,w)$, $Y_{AGZ}(i,w)$.

    Compute $D^2_{AGZ}(i,w) = (\,X_{AGZ}(i,w) - X_i\,)^2 + (\,Y_{AGZ}(i,w) - Y_i\,)^2$.

(1)   If $B_{no\ shift}(i,w) = 1$ (single target, no preclusion, no DGZ shift)

          If $D^2_{AGZ}(i,w) \leq d^2_{iw}$

             Set $B_{defeat\ no\ shift}(i,\ w) = 1$

(2)   Else if $B_{shift}(i,w) = 1$ (single target, no preclusion, DGZ shift)

          If $D^2_{AGZ}(i,w) \leq d^2_{iw}$

             Set $B_{defeat\ shift}(i,\ w) = 1$

(3)   Else if $B_{no\ shift\ aggr}(i,w) = 1$ (aggregate target, no preclusion, no DGZ shift)

          For $j = I_{aggr}(i,w)$, compute $D^2_{AGZ}(j,w) = (X_{AGZ}(j,w) - X_j)^2 + (Y_{AGZ}(j,w) - Y_j)^2$.

          If $D^2_{AGZ}(i,w) \leq d^2_{iw}$

             Set $B_{defeat\ no\ shift\ aggr}(i,\ w) = 1$

          If $D^2_{AGZ}(j,w) \leq d^2_{jw}$

             Set $B_{defeat\ no\ shift\ aggr}(j,\ w) = 1$

(4)   Else if $B_{shift\,aggr}(i,w) = 1$ (aggregate target, no preclusion, DGZ shift)

For $j = I_{aggr}(i,w)$, compute $D^2_{AGZ}(j,w) = (X_{AGZ}(j,w) - X_j)^2 + (Y_{AGZ}(j,w) - Y_j)^2$.

If $D^2_{AGZ}(i,w) \leq d^2_{iw}$

Set $B_{defeat\,shift\,aggr}(i,w) = 1$

If $D^2_{AGZ}(j,w) \leq d^2_{jw}$

Set $B_{defeat\,shift\,aggr}(j,w) = 1$

88.   Loop on all weapons w.

89. Loop on all units i.

Repeat N times. Then calculate the following estimators:

a. $\overline{p}_{prec}(i \mid w, a_i, a\overline{g}g_i) = 1 - \frac{1}{N} \sum B_{WP_k U}(i,w)$

b. $p_{no\,shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) = \frac{1}{N} \sum B_{no\,shift}(i,w)$

c. $p_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) = \frac{1}{N} \sum B_{shift}(i,w)$

d. $\overline{p}_{aggr}(i \mid w, a_i) \cdot \overline{p}_{prec}(i \mid w, a_i, a\overline{g}g_i) = \frac{1}{N} \sum B_{no\,aggr}(i,w)$

e. $p_{engage}(i,\overline{s} \mid w) = p_{no\,shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) = \frac{1}{N} \sum B_{no\,shift}(i,w)$

f. $p_{engage}(i,s \mid w) = p_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) = \frac{1}{N} \sum B_{shift}(i,w)$

g. $p_{no\,shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij}) = \frac{1}{N} \sum B_{no\,shift\,aggr}(i,w)$

h. $p_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij}) = \frac{1}{N} \sum B_{shift\,aggr}(i,w)$

i. $p_{engage}(ij,\overline{s} \mid w) = p_{no\,shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij}) = \frac{1}{N} \sum B_{no\,shift\,aggr}(i,w)$

j. $p_{engage}(ij,s \mid w) = p_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij}) = \frac{1}{N} \sum B_{shift\,aggr}(i,w)$

k. $p_{defeat}(i,\overline{s}, a\overline{g}g_i \mid w, a_i) = \frac{1}{N} \sum B_{defeat\,no\,shift}(i,w)$

l. $p_{defeat}(i,s, a\overline{g}g_i \mid w, a_i) = \frac{1}{N} \sum B_{defeat\,shift}(i,w)$

m. $p_{defeat}(i,\overline{s}, agg_{ij} \mid w, a_{ij}) = \frac{1}{N} \sum B_{defeat\,no\,shift\,aggr}(i,w)$

n. $p_{defeat}(i,s, agg_{ij} \mid w, a_{ij}) = \frac{1}{N} \sum B_{defeat\,shift\,aggr}(i,w)$

o. $p_{defeat}(j,\overline{s}, agg_{ij} \mid w, a_{ij}) = \frac{1}{N} \sum B_{defeat\,no\,shift\,aggr}(j,w)$

p. $p_{defeat}(j,s, agg_{ij} \mid w, a_{ij}) = \frac{1}{N} \sum B_{defeat\,shift\,aggr}(i,w)$

NOTE: $p_{defeat}(i \mid w) = p_{avail}(i) \cdot \left\{ p_{defeat}(i, \overline{s}, a\overline{g}g_i \mid w, a_i) + p_{defeat}(i, s, a\overline{g}g_i \mid w, a_i) \right.$

$\left. + \left\{ p_{defeat}(i, \overline{s}, agg_{ij} \mid w, a_{ij}) + p_{defeat}(i, s, agg_{ij} \mid w, a_{ij}) \right\} \cdot p_{avail}(j) \right\}$

$p_{defeat}(i \mid w) = \frac{1}{N} \sum p_{avail}(i) \cdot \left\{ B_{defeat\ no\ shift}(i, w) + B_{defeat\ shift}(i, w) \right.$

$\left. + \left\{ B_{defeat\ no\ shift\ aggr}(i, w) + B_{defeat\ shift\ aggr}(i, w) \right\} \cdot p_{avail}(j) \right\}$

If units i, j are assumed to be available, then

$p_{defeat}(i \mid w, a_{ij}) = \frac{1}{N} \sum \left\{ B_{defeat\ no\ shift}(i, w) + B_{defeat\ shift}(i, w) \right.$

$\left. + B_{defeat\ no\ shift\ aggr}(i, w) + B_{defeat\ shift\ aggr}(i, w) \right\}$

APPENDIX F

GENERATING REALIZATIONS

Generating Multinormal Pseudorandom Variables

In order to generate unit locations, DGZs, AGZs, etc., it may be necessary to generate pseudorandom variables (realizations of the random variables) from an m-dimensional multinormal probability density function:

$$f(x \mid \mu, \Sigma) = \left[\frac{1}{2\pi} \right]^{-\frac{m}{2}} |\Sigma|^{-\frac{1}{2}} \exp\{ -\tfrac{1}{2} (x - \mu)^T \Sigma^{-1}(x - \mu) \}, \qquad \infty < x < \infty.$$

where $\mu \equiv (\mu_1, \mu_2, ..., \mu_m)^T$ is an m-dimensional known mean vector and Σ is a $(m \times m)$ - dimensional positive definite symmetric variance-covariance matrix. We define a matrix $R \equiv \Sigma^{-1}$. R is referred to as the *precision matrix* of the distribution. μ is assumed known but R is assumed unknown with a multivariate Wishart prior. To generate realizations from this distribution, we assume that the matrix R is given, using a point estimate (the mean) from the current Wishart prior distribution.

The following method is given in Scheuer and Stollar [1962] and summarized in Law and Kelton [1982, pg. 269]. Since Σ is a positive definite symmetric matrix, it can be factored uniquely as $\Sigma = CC^T$, where the $(m \times m)$ matrix C is lower triangular. Let c_{ij} denote the (i, j)th element of C. The algorithm for generating X_1, X_2, ..., X_m as given in Appendix A is:

1. Generate Z_1, Z_2, ..., Z_m as iid $N(0,1)$ random variables.

2. For $i = 1, ..., m$, let $X_i = \mu_i + \sum_{j=1}^{m} c_{ij} Z_j$.

Thus each X_i is a linear combination of the Z_i's. The Y_i's can be generated using the same algorithm with different c_{ij}'s.

Generating Nuclear Laydown Realizations for FORCEM

The objective of this section is to specify a means for generating a nuclear laydown realization for input to FORCEM, given the defeated / not defeated status of each FORCEM unit from the experimental design. Several criteria will be followed:

1. Each division (FORCEM unit) will have at least the required number of potential target units defeated that are required to defeat the division ($n_{min\ defeat}$).

2. The units will be ranked in order of target priority. This permits the weapon allocation to be directed against the units of highest priority.

3. For each FORCEM division that must be defeated, the first $n_{min\ defeat}$ units in order of decreasing $p_{defeat}(i)$ will have an $p_{avail} = 1$; in other words, they will assumed to be acquired, retained, and available for fire planning. They will also have perfect weapons reliability (no failures) and AGZs will be generated randomly for each available weapon in order of weapon preference. The first weapon that yields an AGZ with the target coverage to defeat the unit will be selected for firing against that unit. The remaining units will be available randomly according to p_{avail} for each unit. weapons may fail randomly, and AGZs are not forced to cover the target. This insures that at least $n_{min\ defeat}$ are available for fire, and those units which are to be struck are those most likely to be defeated.

4. For each FORCEM division that must *not* be defeated, the first $(m - n_{min\ defeat} + 1)$ units in order of increasing $p_{defeat}(i)$ will have an $p_{avail} = 0$; in other words, they will assumed to be not acquired or otherwise unavailable for fire planning. This insures that at least $(m - n_{min\ defeat} + 1)$ are never engaged, so the division cannot be defeated. The remaining units will be available randomly according to p_{avail} for each unit, weapons may fail randomly, and AGZs are not forced to cover the target.

In order to keep the algorithm manageable, the generation of a FORCEM realization will most likely have to be done in parts, probably Blue corps sectors for Blue against Red and Red army sectors for Red against Blue. Within each sector, perform the following algorithm:

ALGORITHM:

For each division (FORCEM unit) $j = 1$ to $n_{division}$ to be defeated in order of division target priority

    Generate actual and perceived locations for all potential target units

    Order the unit $p_{defeat}(i)$ values in decreasing order. Set $p_{min\ defeat} =$ the $n_{min\ defeat}$th probability in order

    Order the units by decreasing target priority

[1]   Let $N_{defeat} = 0$

For each unit in division j, $i = 1$ to m_j in order

    If $p_{defeat}(i) < p_{min\ defeat}$,

        Draw U $\sim$ Uniform(0,1)

        If U $> p_{avail}$, go to next i

    Else for each weapon $w = 1$ to nw in order by target preference for unit i

        If weapon w is *not* available within range, go to next w

        Else determine the DGZ for unit i with weapon w based on preclusion and

            aggregation

        If DGZ precluded, go to next w

        Else if $p_{defeat}(i) < p_{min\ defeat}$,

            Mark weapon w as expended

            Draw U $\sim$ Uniform(0,1)

            If U $>$ weapon system reliability, go to next i (a dud was fired)

            Else generate and store AGZ for unit i using w (successful detonation)

            If AGZ to actual location distance $\leq d_{iw}$, let $N_{defeat} = N_{defeat} + 1$

            Exit loop on w

        Else ($p_{defeat}(i) \geq p_{min\ defeat}$)

            Generate an AGZ for weapon w

            If AGZ to actual location distance $> d_{iw}$, go to next w (try again)

            Else store AGZ for unit i

            Let $N_{defeat} = N_{defeat} + 1$

            Mark weapon w as expended

            Exit loop on w (go to next i)

        End loop on w

    End loop on i

If $N_{defeat} < n_{min\ defeat}$, set $p_{min\ defeat} = $ the $(2 \cdot n_{min\ defeat})$th probability in order,

    destroy all AGZs, restore weapons expended and go to [1]

End loop on j

For each division (FORCEM unit) $j = 1$ to $n_{division}$ *not* to be defeated, in order of division target

    priority

    Generate actual and perceived locations for all potential target units

    Order the unit $p_{defeat}(i)$ values in increasing order. Set $p_{min\ defeat} = $ the

        $(m - n_{min\ defeat} + 1)$th probability in order

    Order the units by decreasing target priority

For each unit in division j, $i = 1$ to m_j in order

    If $p_{defeat}(i) < p_{min\ defeat}$, go to next i (unit not available for engagement)

    Else draw $U \sim$ Uniform(0,1)

    If $U > p_{avail}$, go to next i (unit not available for engagement)

    Else for each weapon $w = 1$ to nw in order by target preference for unit i

        If weapon w is *not* available within range, go to next w

        Else determine the DGZ for unit i with weapon w based on preclusion and aggregation

        If DGZ precluded, go to next w

        Mark weapon w as expended

        Draw $U \sim$ Uniform(0,1)

        If $U >$ weapon system reliability, go to next i (a dud was fired)

        Else generate and store an AGZ for weapon w and unit i (successful detonation)

    End loop on w

  End loop on i

End loop on j

APPENDIX G

EXAMPLE OF AN ANALYTIC SOLUTION

Section I - UPDATING THE MULTINORMAL DISTRIBUTION

Suppose that there is a simple template that consists of only two units, #1 and #2. Both units are mechanized infantry units consisting of personnel in APCs, with a target unit radius of 500 m. The template has unit #1 located at coordinates (0, 0) and unit #2 located at coordinates (0. 1100). When the template is placed on a map and oriented correctly, there are two nearby preclusion areas whose local (X, Y) coordinates translate into (1000, 1000) for preclusion area #1 and (500, 1600) for preclusion area #2.

There is no prior information available on the variance-covariance matrix Σ for this templated force, so a non-informative prior will be used to update Σ.

Four experiments are run where the units are arrayed using the NUFAM-GAP system. The following data are obtained:

$$\text{Expt. \#1:} \quad \begin{bmatrix} x_1 = -145 & y_1 = -275 \\ x_2 = 1770 & y_2 = 840 \end{bmatrix}$$

$$\text{Expt. \#2:} \quad \begin{bmatrix} x_1 = 1080 & y_1 = -85 \\ x_2 = 3020 & y_2 = 1035 \end{bmatrix}$$

$$\text{Expt. \#3:} \quad \begin{bmatrix} x_1 = -1640 & y_1 = 40 \\ x_2 = -1850 & y_2 = 1305 \end{bmatrix}$$

$$\text{Expt. \#4:} \quad \begin{bmatrix} x_1 = 3480 & y_1 = 275 \\ x_2 = 555 & y_2 = 1315 \end{bmatrix}$$

Updating the X-dimension Variance-Covariance Matrix Σ_x:

The precision matrix R_x of the multinormal distribution for X_1, X_2 has a prior Wishart distribution (α, τ_x^{-1}) where τ_x is the precision matrix of the Wishart distribution. The noninformative values of α and τ_x are:

$$\alpha = 0 \qquad \tau_x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From Appendix C, we know that the posterior distribution of R_x is distributed as Wishart ($\alpha + n$, τ_x^{*-1}) where $\tau_x^* = \tau_x + \sum_{i=1}^{n} (x_i - m_x)(x_i - m_x)^t$.

We begin by evaluating ($x_i - m_x$), recalling that $m_x = (0, 0)^t$. Thus $n = 4$ and denoting

$$x_i - m_x \equiv \begin{bmatrix} x_{1i} - m_{x1} \\ x_{2i} - m_{x2} \end{bmatrix}$$

$$x_1 - m_x = \begin{bmatrix} -175 \\ 1770 \end{bmatrix} \qquad x_2 - m_x = \begin{bmatrix} 1080 \\ 3020 \end{bmatrix} \qquad x_3 - m_x = \begin{bmatrix} -1640 \\ -1850 \end{bmatrix} \qquad x_4 - m_x = \begin{bmatrix} 3480 \\ 555 \end{bmatrix}$$

$$\tau_x^* = \tau_x + \sum_{i=1}^{n} (x_i - m_x)(x_i - m_x)^t$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{n}(x_{1i} - m_{x1})^2 & \sum_{i=1}^{n}(x_{1i} - m_{x1})(x_{2i} - m_{x2}) \\ \sum_{i=1}^{n}(x_{1i} - m_{x1})(x_{2i} - m_{x2}) & \sum_{i=1}^{n}(x_{2i} - m_{x2})^2 \end{bmatrix}$$

$$\tau_x^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 15{,}987{,}425 & 7{,}970{,}350 \\ 7{,}970{,}350 & 15{,}983{,}825 \end{bmatrix} = \begin{bmatrix} 15{,}987{,}425 & 7{,}970{,}350 \\ 7{,}970{,}350 & 15{,}983{,}825 \end{bmatrix}$$

For the analytic model, we assume the variables X_1, X_2 to be distributed as multinormal (μ_x, Σ_x) where μ_x is known (the templated mean locations) and $\Sigma_x = E[R_x^{-1}]$. From the Wishart distribution, we know that $E[R_x] = (\alpha + n)\tau_x^{*-1}$. Thus $\Sigma_x = E[R_x^{-1}] = E[\{(\alpha + n)\tau_x^{*-1}\}^{-1}] = \frac{1}{\alpha + n}\tau_x^*$.

$$\Sigma_x = \frac{1}{4}\begin{bmatrix} 15{,}987{,}425 & 7{,}970{,}350 \\ 7{,}970{,}350 & 15{,}983{,}825 \end{bmatrix} = \begin{bmatrix} 3{,}996{,}856 & 1{,}992{,}588 \\ 1{,}992{,}588 & 3{,}995{,}956 \end{bmatrix}.$$

Thus $\sigma_{x1}^2 \equiv \text{Var}[\ X_1\ ] = 3{,}996{,}856;\ \sigma_{x2}^2 \equiv \text{Var}[\ X_2\ ] = 3{,}995{,}956$ and

$\sigma_{x12} \equiv \text{Cov}[\ X_1,\ X_2\ ] = 1{,}992{,}588$, with $\rho_{x12} \equiv \dfrac{\sigma_{x12}}{\sigma_{x1}\sigma_{x2}} = 0.499 \doteq 0.5$.

For convenience, we round off as follows:

$\sigma_{x1} = 1{,}999 \doteq 2{,}000;\ \sigma_{x2} = 1{,}999 \doteq 2{,}000$ and

$\sigma_{x12} = 0.499\,(1{,}999)\,(1{,}999) \doteq 0.5\,(2{,}000)\,(2{,}000)$.

Updating the Y-dimension Variance-Covariance Matrix Σ_y:

The precision matrix $\mathbf{R}_y$ of the multinormal distribution for Y_1, Y_2 has a prior Wishart distribution $(\ \alpha,\ \tau_y{}^{-1}\ )$ where τ_y is the precision matrix of the Wishart distribution. The noninformative values of α and τ_x are:

$$\alpha = 0 \qquad \tau_x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

From Appendix C, we know that the posterior distribution of $\mathbf{R}_y$ is distributed as Wishart $(\ \alpha + n,\ \tau_y{}^{*-1}\ )$ where $\tau_y{}^* = \tau_y + \sum\limits_{i=1}^{n} (\ y_i - \mathbf{m}_y\ )(\ y_i - \mathbf{m}_y\ )^t$.

We begin by evaluating $(\ y_i - \mathbf{m}_y\ )$, recalling that $\mathbf{m}_y = (\ 0,\ 1100\ )^t$. Thus $n = 4$ and

$$y_1 - \mathbf{m}_y = \begin{bmatrix} -275 \\ -260 \end{bmatrix} \qquad y_2 - \mathbf{m}_y = \begin{bmatrix} -85 \\ -65 \end{bmatrix} \qquad y_3 - \mathbf{m}_y = \begin{bmatrix} 40 \\ 205 \end{bmatrix} \qquad y_4 - \mathbf{m}_y = \begin{bmatrix} 275 \\ 215 \end{bmatrix}$$

$$\tau_y{}^* = \tau_y + \sum_{i=1}^{n} (\ y_i - \mathbf{m}_y\ )(\ y_i - \mathbf{m}_y\ )^t$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sum\limits_{i=1}^{n} (y_{1i} - m_{y1})^2 & \sum\limits_{i=1}^{n} (y_{1i} - m_{y1})(y_{2i} - m_{y2}) \\ \sum\limits_{i=1}^{n} (y_{1i} - m_{y1})(y_{2i} - m_{y2}) & \sum\limits_{i=1}^{n} (y_{2i} - m_{y2})^2 \end{bmatrix}$$

$$\tau_y{}^* = \begin{bmatrix} 160{,}075 & 144{,}350 \\ 144{,}350 & 160{,}075 \end{bmatrix}$$

For the analytic model, we assume the variables Y_1, Y_2 to be distributed as multinormal $(\ \mu_y,\ \Sigma_y\ )$ where μ_y is known (the templated mean locations) and $\Sigma_y = \text{E}[\ \mathbf{R}_y{}^{-1}\ ] = \text{E}[\ \{(\alpha + n)\ \tau_y{}^{*-1}\}^{-1}\ ] = \dfrac{1}{\alpha + n}\ \tau_y{}^*$. Thus

$$\Sigma_y = \frac{1}{4} \begin{bmatrix} 160,075 & 144,350 \\ 144,350 & 160,075 \end{bmatrix} = \begin{bmatrix} 40,019 & 36,088 \\ 36,088 & 40,019 \end{bmatrix}$$

Thus $\sigma_{y1}^2 \equiv \text{Var}[\ Y_1\ ] = 40,019 = \sigma_{y2}^2 \equiv \text{Var}[\ Y_2\ ]$

$\sigma_{y12} \equiv \text{Cov}[\ Y_1,\ Y_2\ ] = 36,088$, with $\rho_{y12} \equiv \frac{\sigma_{y12}}{\sigma_{y1}\sigma_{y2}} = 0.902 \doteq 0.9$.

For convenience, we round off as follows:

$\sigma_{y1} = 200.0 \doteq 200;\ \sigma_{y2} = 200.0 \doteq 200$ and

$\sigma_{y12} = 0.902(200.0)(200.0) \doteq 0.9(200)(200)$.

Section II - SINGLE UNIT , SINGLE PRECLUSION AREA

Given information

Suppose we have a unit with the following characteristics:

Radius: 500 m.

Target elements: Personnel in APCs, Transmission Factor (TF) = 0.7

Mean location (μ_x, μ_y) = (0, 0)

Location variance: $\sigma_{x1}^2 = 2000^2$ m^2; $\sigma_{y1}^2 = 200^2$ m^2

$P_{avail}(1) = 1.0$ (for convenience, we assume that it is available)

Target location error (TLE): $\mu_{tx1} = \mu_{ty1} = 0$; $\sigma_{tx1}^2 = 75^2$ m^2, $\sigma_{ty1}^2 = 100^2$ m^2

We also have the following weapon characteristics:

Weapon 1:   Yield: 1 kt   CEP: 150m

Weapon 2:   Yield: 10 kt   CEP: 100m

From the CEP, we can compute the distribution of the shift due to weapon accuracy:

Weapon type 1: $\mu_{cx1} = \mu_{cy1} = 0$; $\sigma_{cx1} = \sigma_{cy1} = \frac{150}{.67} = 225$ m

Weapon type 2: $\mu_{cx1} = \mu_{cy1} = 0$; $\sigma_{cx1} = \sigma_{cy1} = \frac{100}{.67} = 150$ m

We also have a required target coverage (target defeat criterion) of 30% coverage with IT radiation effects against personnel (3000 rad).

There is a preclusion area (#1) with a radius of 1000 m. located at coordinates (x_{P_1}, y_{P_1}) = (1000, 1000).

The preclusion area requirement is assumed for this example to avoid an exposure of more than 75 rad to exposed individuals, with a safety buffer of twice the weapon CEP.

Computing the necessary constants

From the above information, we can compute the required constants:

$r_{T_1} = 500$ m (given)

Using the following (unclassified) formulas for converting dose to distance, we can find the r_W ranges.

3000 rad inside dose (0.7 TF) $= \dfrac{3000}{.7} = 4286$ outside dose

For a 1 kt yield,

$D = \exp\{ 14\ e^{-8.6 \times 10^{-4}\ R} \}$, where D = (outside) dose in rad and R is the range in meters.

From this, a 4286 rad dose is achieved at 600 m and a 75 rad dose at 1370 m.

For a 10 kt yield,

$D = \exp\{ 16.4\ e^{-6.8 \times 10^{-4}\ R} \}$, where D = (outside) dose in rad and R is the range in meters.

From this, a 4286 rad dose is achieved at 990 m and a 75 rad dose at 1960 m.

This yields $r_{W_1} = 600$ m. and $r_{W_2} = 990$ m.

For preclusion, r_{W_1}(preclusion) $= r_{W_1}$(75 rad) $+ 2(CEP_1) = 1370 + 2(150) = 1670$;

r_{W_2}(preclusion) $= r_{W_2}$(75 rad) $+ 2(CEP_2) = 1960 + 2(100) = 2160$.

Thus the r_{WP_1} values are: $r_{W_1 P_1} = 1670 + 1000 = 2670$ m; $r_{W_2 P_1} = 2160 + 1000 = 3160$ m.

30% of a 500 m radius target area is covered by a 1 kt weapon ($r_{W_1} = 600$ m) at a distance of 925 m. 30% of a 500 m radius target area is covered by a 10 kt weapon ($r_{W_2} = 990$ m) at a distance of 1355 m. Thus $d_{1\,W_1} = 925$ m and $d_{1\,W_2} = 1355$ m.

From the r_W(preclusion), r_{P_1} and $d_{1\,W}$ values, we can compute r_{WP_1} and $r_{WP_1 U_1}$ for both weapons.

$r_{W_1 P_1} = 1670 + 1000 = 2670$ m; $r_{W_2 P_1} = 2160 + 1000 = 3160$ m.

$r_{W_1 P_1 U_1} = 2670 - 925 = 1745$ m; $r_{W_2 P_1 U_1} = 3160 - 1355 = 1805$ m.

Computing the distributions from target unit #1 to preclusion area #1

We can compute the parameters of the important distributions from the target and preclusion area information.

$\mu_{x1\,p_1} \equiv (\mu_{x1} + \mu_{tx1}) - (x_{P_1} + \mu_{tx\,p_1}) = (0 + 0) - (1000 + 0) = -1{,}000$

$\mu_{y1\,p_1} \equiv (\mu_{y1} + \mu_{ty1}) - (y_{P_1} + \mu_{ty\,p_1}) = (0 + 0) - (1000 + 0) = -1{,}000$

$\sigma^2_{x1\,p_1} \equiv (\sigma^2_{x1} + \sigma^2_{tx1}) + (\sigma^2_{P_1} + \sigma^2_{tx\,p_1}) = (2000^2 + 75^2) + (0 + 0) = 4{,}005{,}625$

$\sigma^2_{y1\,p_1} \equiv (\sigma^2_{y1} + \sigma^2_{ty1}) + (\sigma^2_{P_1} + \sigma^2_{ty\,p_1}) = (200^2 + 100^2) + (0 + 0) = 50{,}000$

Let $\mu_{Ip_1} \equiv \mu_{x I p_1}^2 + \mu_{y I p_1}^2 + \sigma_{x I p_1}^2 + \sigma_{y I p_1}^2$ and

$\quad \sigma_{Ip_1}^2 \equiv 2\sigma_{x I p_1}^2(\sigma_{x I p_1}^2 + 2\mu_{x I p_1}^2) + 2\sigma_{y I p_1}^2(\sigma_{y I p_1}^2 + 2\mu_{y I p_1}^2)$

Then

$\quad D_{Ip_1}^2 \sim N[\mu_{Ip_1}, \sigma_{Ip_1}^2]$ or $D_{Ip_1}^2 \sim N[6{,}055{,}625;\ 4.832 \times 10^{13}]$.

$\quad \mu_{Ip_1} = 6{,}055{,}625$

$\quad \sigma_{Ip_1} = 6{,}951{,}084$

Computing the probabilities of preclusion and shift (unit #1 from preclusion area #1)

$$P_{no\ shift}(1\,|\,w_1, a_1, a\overline{g}g_1) \quad = P[\,D_{Ip_1}^2 > r_{W_1 P_1}^2\,] = P[\,Z > \frac{2670^2 - \mu_{Ip_1}}{\sigma_{Ip_1}}\,] = P[\,Z > 0.154\,]$$
$$= 0.439$$

$$P_{no\ shift}(1\,|\,w_2, a_1, a\overline{g}g_1) \quad = P[\,D_{Ip_1}^2 > r_{W_2 P_1}^2\,] = P[\,Z > \frac{3160^2 - \mu_{Ip_1}}{\sigma_{Ip_1}}\,] = P[\,Z > 0.565\,]$$
$$= 0.286$$

$$\overline{p}_{prec}(1\,|\,w_1, a_1, a\overline{g}g_1) = P[\,D_{Ip_1}^2 > r_{W_1 P_1 U_1}^2\,] = P[\,Z > \frac{1745^2 - \mu_{Ip_1}}{\sigma_{Ip_1}}\,] = P[\,Z > -0.433\,]$$
$$= 0.668$$

$$\overline{p}_{prec}(1\,|\,w_2, a_1, a\overline{g}g_1) = P[\,D_{Ip_1}^2 > r_{W_2 P_1 U_1}^2\,] = P[\,Z > \frac{1805^2 - \mu_{Ip_1}}{\sigma_{Ip_1}}\,] = P[\,Z > -0.402\,]$$
$$= 0.656$$

Since $p_{shift}(1\,|\,w, a_1, a\overline{g}g_1) = \overline{p}_{prec}(1\,|\,w, a_1, a\overline{g}g_1) - p_{no\ shift}(1\,|\,w, a_1, a\overline{g}g_1)$,

$\quad p_{shift}(1\,|\,w_1, a_1, a\overline{g}g_1) = 0.668 - 0.439 = 0.229$

$\quad p_{shift}(1\,|\,w_2, a_1, a\overline{g}g_1) = 0.656 - 0.286 = 0.370$

Computing the AGZ and conditional defeat distributions

Let AGZ_1 denote the AGZ from firing the weapon of type 1 and AGZ_2 denote the AGZ from firing the weapon of type 2.

(1) *No DGZ shift, Weapon type 1:*

$\quad X_{AGZ_1} = X_1 + TX_1 + CX_1$; $Y_{AGZ_1} = Y_1 + TY_1 + CY_1$. Thus

$\quad X_{AGZ_1} - X_1 = TX_1 + CX_1$; $Y_{AGZ_1} - Y_1 = TY_1 + CY_1$.

Let $\mu_{x I AGZ_1} \equiv E[X_{AGZ_1} - X_1] = \mu_{tx1} + \mu_{cx1}$; $\mu_{y I AGZ_1} \equiv E[Y_{AGZ_1} - Y_1] = \mu_{ty1} + \mu_{cy1}$.

$\quad \sigma_{x I AGZ_1}^2 \equiv Var[X_{AGZ_1} - X_1] = \sigma_{tx1}^2 + \sigma_{cx1}^2$; $\sigma_{y I AGZ_1}^2 \equiv Var[Y_{AGZ_1} - Y_1] = \sigma_{ty1}^2 + \sigma_{cy1}^2$.

$$\mu_{1\,AGZ_1} \equiv \mu_{x\,1\,AGZ_1}^2 + \mu_{y\,1\,AGZ_1}^2 + \sigma_{x\,1\,AGZ_1}^2 + \sigma_{y\,1\,AGZ_1}^2$$

$$\sigma_{1\,AGZ_1}^2 \equiv 2\sigma_{x\,1\,AGZ_1}^2(\sigma_{x\,1\,AGZ_1}^2 + 2\mu_{x\,1\,AGZ_1}^2) + 2\sigma_{y\,1\,AGZ_1}^2(\sigma_{y\,1\,AGZ_1}^2 + 2\mu_{y\,1\,AGZ_1}^2)$$

Then the squared distance between the AGZ for weapon of type w and the unit 1, $D^2_{1\,AGZ_w}$, is:

$$D^2_{1\,AGZ_1} = D^2_{x\,1\,AGZ_1} + D^2_{y\,1\,AGZ_1} \sim N[\mu_{1\,AGZ_1}, \sigma^2_{1\,AGZ_1}]$$

Evaluating the terms,

$$\mu_{x\,1\,AGZ_1} = 0 \; ; \; \mu_{y\,1\,AGZ_1} = 0$$

$$\sigma^2_{x\,1\,AGZ_1} = 75^2 + 225^2 = 56{,}250; \; \sigma^2_{y\,1\,AGZ_1} = 100^2 + 225^2 = 60{,}625.$$

$$\mu_{1\,AGZ_1} = 0 + 0 + 56{,}250 + 60{,}625 = 116{,}875$$

$$\sigma^2_{1\,AGZ_1} = 2(56{,}250)^2 + 2(60{,}625)^2 = 13{,}678{,}906{,}250$$

$$\sigma_{1\,AGZ_1} = 116{,}957$$

The conditional probability of defeat given weapon type 1 and no DGZ shift is:

$$P_{defeat}(1\,|\,w_1,\,a_1,\,\overline{s},\,a\overline{g}g_1) \; = P[\,D^2_{1\,AGZ_1} \leq d^2_{1\,W_1}\,] = P[\,Z \leq \frac{925^2 - \mu_{1\,AGZ_1}}{\sigma_{1\,AGZ_1}}\,] = P[\,Z \leq 6.32\,]$$
$$\doteq 1.0$$

(2) *No DGZ shift, Weapon type 2:*

The squared distance between the AGZ for weapon of type #2 and the unit 1, $D^2_{1\,AGZ_2}$, is evaluated as before.

$$\mu_{x\,1\,AGZ_2} = 0 \; ; \; \mu_{y\,1\,AGZ_2} = 0$$

$$\sigma^2_{x\,1\,AGZ_2} = 75^2 + 150^2 = 28{,}125; \; \sigma^2_{y\,1\,AGZ_2} = 100^2 + 150^2 = 32{,}500$$

$$\mu_{1\,AGZ_2} = 60{,}625$$

$$\sigma^2_{1\,AGZ_2} = 3{,}694{,}531{,}250$$

$$\sigma_{1\,AGZ_2} = 60{,}783$$

The conditional probability of defeat given weapon type 1 and no DGZ shift is:

$$P_{defeat}(1\,|\,w_2,\,a_1,\,\overline{s},\,a\overline{g}g_1) = P[\,D^2_{1\,AGZ_2} \leq d^2_{1\,W_2}\,] = P[\,Z \leq \frac{1355^2 - \mu_{1\,AGZ_2}}{\sigma_{1\,AGZ_2}}\,]$$
$$= P[\,Z \leq 29.209\,] \doteq 1.0$$

(3) *DGZ shift, Weapon type 1:*

Recall that $r_{W_1 P_1} = 2670$ and $E[\,D^2_{1\,P_1}] = 6,055,625$. Thus

$$\beta_1 = \frac{r_{W_1 P_1}}{\sqrt{E[\,D^2_{1\,P_1}]}} = 1.085$$

The shifted DGZ coordinates are:

$$X_{AGZ^s_1} = \beta\,X_{1\,L} + (\,1 - \beta\,)\,x_{P_1} = \beta\,(\,X_1 + TX_1\,) + (\,1 - \beta\,)\,x_{P_1}$$

$$Y_{AGZ^s_1} = \beta\,Y_{1\,L} + (\,1 - \beta\,)\,y_{P_1} = \beta\,(\,Y_1 + TY_1\,) + (\,1 - \beta\,)\,y_{P_1}$$

Thus

$$X_{AGZ^s_1} - X_1 = (\,\beta - 1\,)\,X_1 + \beta\,TX_1 + (\,1 - \beta\,)\,x_{P_1} + CX_1 \text{ and}$$

$$Y_{AGZ^s_1} - Y_1 = (\,\beta - 1\,)\,Y_1 + \beta\,TY_1 + (\,1 - \beta\,)\,y_{P_1} + CY_1.$$

Let

$$\mu_{x1\,AGZ^s_1} \equiv E[X_{AGZ^s_1} - X_1] = (\beta - 1)\mu_{x1} + \beta\mu_{tx1} + (\,1 - \beta\,)\,x_{P_1} + \mu_{cx1}$$

$$\mu_{y1\,AGZ^s_1} \equiv E[Y_{AGZ^s_1} - Y_1] = (\beta - 1)\mu_{y1} + \beta\mu_{ty1} + (\,1 - \beta\,)\,y_{P_1} + \mu_{cy1}$$

$$\sigma^2_{x1\,AGZ^s_1} \equiv \text{Var}[X_{AGZ^s_1} - X_1] = (\beta - 1)\sigma^2_{x1} + \beta\sigma^2_{tx1} + \sigma^2_{cx1}$$

$$\sigma^2_{y1\,AGZ^s_1} \equiv \text{Var}[Y_{AGZ^s_1} - Y_1] = (\beta - 1)\sigma^2_{y1} + \beta\sigma^2_{ty1} + \sigma^2_{cy1}$$

$$\mu_{1\,AGZ^s_1} \equiv \mu^2_{x1\,AGZ^s_1} + \mu^2_{y1\,AGZ^s_1} + \sigma^2_{x1\,AGZ^s_1} + \sigma^2_{y1\,AGZ^s_1}$$

$$\sigma^2_{1\,AGZ^s_1} \equiv 2\sigma^2_{x1\,AGZ^s_1}(\sigma^2_{x1\,AGZ^s_1} + 2\mu^2_{x1\,AGZ^s_1}) + 2\sigma^2_{y1\,AGZ^s_1}(\sigma^2_{y1\,AGZ^s_1} + 2\mu^2_{y1\,AGZ^s_1})$$

Then the squared distance between the AGZ (based on a shifted DGZ) for weapon of type #1 and the unit 1, $D^2_{1\,AGZ^s_1}$, is:

$$D^2_{1\,AGZ^s_1} = D^2_{x1\,AGZ^s_1} + D^2_{y1\,AGZ^s_1} \sim N[\mu_{1\,AGZ^s_1},\,\sigma^2_{1\,AGZ^s_1}]$$

Evaluating the terms,

$$\mu_{x1\,AGZ^s_1} = 0.085(0) + 1.085(0) - 0.085(1000) + 0 = -85 = \mu_{y1\,AGZ^s_1}$$

$$\sigma^2_{x1\,AGZ^s_1} = 0.085(2000^2) + 1.085(75^2) + 225^2 = 396,749$$

$$\sigma^2_{y1\,AGZ^s_1} = 0.085(200^2) + 1.085(100^2) + 225^2 = 64,875$$

$$\mu_{1\,AGZ^s_1} = 2(-85)^2 + 396,749 + 64,875 = 76,075$$

$$\sigma^2_{1\,AGZ^s_1} = 2(396,749)(396,749 + (2)(-85)^2) + 2(64,875)(64,875 + (2)(-85)^2) = 336,578,874,208$$

The conditional probability of defeat given weapon type 1 and a DGZ shift is:

$$P_{defeat}(1 \mid w_1, a_1, s, a\overline{g}g_1) = P[\,D^2_{1\,AGZ^s_1} \leq d^2_{1\,W_1}] = P[\,Z \leq \frac{925^2 - \mu_{1\,AGZ^s_1}}{\sigma_{1\,AGZ^s_1}}] = P[\,Z \leq 0.654\,]$$

$$= 0.744$$

(4) *DGZ shift, Weapon type 2:*

Recall that $r_{W_2 P_1} = 3160$ and $E[\, D^2_{1\, P_1}] = 6,055,625$. Thus

$$\beta_2 = \frac{r_{W_2 P_1}}{\sqrt{E[\, D^2_{1\, P_1}]}} = 1.284$$

The squared distance between the AGZ (based on a shifted DGZ) for weapon of type 2 and the unit 1, $D^2_{1\, AGZ_w}$, is evaluated as before:

$$D^2_{1\, AGZ^s_2} = D^2_{x\, 1\, AGZ^s_2} + D^2_{y\, 1\, AGZ^s_2} \sim N[\mu_{1\, AGZ^s_2}, \sigma^2_{1\, AGZ^s_2}]$$

Evaluating the terms,

$$\mu_{x\, 1\, AGZ^s_2} = -0.284(1000) = -284 = \mu_{y\, 1\, AGZ^s_2}$$
$$\sigma^2_{x\, 1\, AGZ^s_2} = 0.284(2000^2) + 1.284(75^2) + 150^2 = 1,160,227$$
$$\sigma^2_{y\, 1\, AGZ^s_2} = 0.284(200^2) + 1.284(100^2) + 150^2 = 46,706$$
$$\mu_{1\, AGZ^s_2} = 1,374,388$$
$$\sigma^2_{1\, AGZ^s_2} = 3.116 \times 10^{12}$$

The conditional probability of defeat given weapon type 2 and a DGZ shift is:

$$
\begin{aligned}
P_{defeat}(1 \mid w_2, a_1, s, a\overline{g}g_1) \quad &= P[\, D^2_{1\, AGZ^s_2} \le d^2_{1\, W_2}\,] = P[\, Z \le \frac{1355^2 - \mu_{1\, AGZ^s_2}}{\sigma_{1\, AGZ^s_2}}\,] = P[\, Z \le 0.262\,] \\
&= 0.738
\end{aligned}
$$

Computing the defeat distributions for target unit #1 as an only target

Suppose that $p_{round}(\, 1, w_1\,) = 0.75$, $p_{round}(\, 1, w_2\,) = 0.25$. Since p_{avail} was assumed to be 1.0 and $\overline{p}_{aggr}(1 \mid w_1, a_1) = \overline{p}_{aggr}(1 \mid w_2, a_1) = 1.0$ as there are no other units in this example, $p_{engage}(1, \overline{s} \mid w) = p_{no\ shift}(1 \mid w, a_1, a\overline{g}g_1)$ and $p_{engage}(1, s \mid w) = p_{shift}(1 \mid w, a_1, a\overline{g}g_1)$ for $w = 1, 2$. Thus

$$
\begin{aligned}
P_{defeat}(\, 1\,) = {}& p_{defeat}(1 \mid w_1, a_1, \overline{s}, a\overline{g}g_1) \cdot p_{no\ shift}(1 \mid w_1, a_1, a\overline{g}g_1) \cdot p_{round}(\, 1, w_1\,) \\
&+ p_{defeat}(1 \mid w_1, a_1, s, a\overline{g}g_1) \cdot p_{shift}(1 \mid w_1, a_1, a\overline{g}g_1) \cdot p_{round}(\, 1, w_1\,) \\
&+ p_{defeat}(1 \mid w_2, a_1, \overline{s}, a\overline{g}g_1) \cdot p_{no\ shift}(1 \mid w_2, a_1, a\overline{g}g_1) \cdot p_{round}(\, 1, w_2\,) \\
&+ p_{defeat}(1 \mid w_2, a_1, s, a\overline{g}g_1) \cdot p_{shift}(1 \mid w_2, a_1, a\overline{g}g_1) \cdot p_{round}(\, 1, w_2\,) \\
={}& 1.00 \cdot 0.439 \cdot 0.75 + 0.744 \cdot 0.229 \cdot 0.75 + 1.00 \cdot 0.286 \cdot 0.25 + 0.738 \cdot 0.370 \cdot 0.25 \\
={}& 0.622
\end{aligned}
$$

NOTE: The probabilities of defeating unit #1 will be recomputed in the next section, when an additional preclusion area and an additional target unit are added to the example.

Section III - SECOND UNIT, SECOND PRECLUSION AREA

Given information

Suppose we have a second unit with the following characteristics:

Radius: 500 m.

Target elements: Personnel in APCs, Transmission Factor (TF) = 0.7

Mean location (μ_{x2}, μ_{y2}) = (0, 1100)

Location variance: $\sigma_{x2}^2 = 2000^2$ m^2; $\sigma_{y2}^2 = 200^2$ m^2

Since units #1 and #2 are jointly distributed as multinormal, we must also define covariances.

Cov[X_1, X_2] $\equiv \sigma_{x12} = \rho_{x12}\sigma_{x1}\sigma_{x2} = 0.5$ (2000)(2000) = 2,000,000

Cov[Y_1, Y_2] $\equiv \sigma_{y12} = \rho_{y12}\sigma_{y1}\sigma_{y2} = 0.9$ (200)(200) = 36,000

$P_{avail}(2) = 1.0$ (for convenience, we assume that it is available)

Target location error (TLE): $\mu_{tx2} = \mu_{ty2} = 0$; $\sigma_{tx2}^2 = 75^2$ m^2, $\sigma_{ty2}^2 = 100^2$ m^2

We also have the same weapon characteristics as before and the same required target coverage (target defeat criterion) of 30% coverage with IT radiation effects against personnel (3000 rad).

There is a preclusion area (#2) with a radius of 500 m. located at coordinates (x_{P_2}, y_{P_2}) = (500. 1600). The preclusion area requirement is the same as in the previous part of this example.

Computing the necessary constants

From the above information, we can compute the required constants:

$r_{T_2} = 500$ m (given)

Since the weapon and target types are the same, $r_{W_1} = 600$ m. and $r_{W_2} = 990$ m.

For preclusion, r_{W_1}(preclusion) $= 1370 + 2(150) = 1670$; r_{W_2}(preclusion) $= 1960 + 2(100) = 2160$.

However, the radius of preclusion area #2 is different. So the r_{WP_2} values are:

$r_{W_1P_2} = 1670 + 500 = 2170$ m; $r_{W_2P_2} = 2160 + 500 = 2660$ m.

30% of a 500 m radius target area is covered by a 1 kt weapon (r_{W_1} = 600 m) at a distance of 925 m, and 30% of a 500 m radius target area is covered by a 10 kt weapon (r_{W_2} = 990 m) at a distance of 1355 m as before. Thus

$$d_{2W_1} = 925 \text{ m and } d_{2W_2} = 1355 \text{ m}.$$

From the r_W(preclusion), r_{P_2} and d_{2W} values, we can compute r_{WP_2} and $r_{WP_2U_2}$ for both weapons.

$$r_{W_1P_2} = 1670 + 500 = 2170 \text{ m}; \; r_{W_2P_2} = 2160 + 500 = 2660 \text{ m}.$$

$$r_{W_1P_2U_2} = 2170 - 925 = 1245 \text{ m}; \; r_{W_2P_2U_2} = 2660 - 1355 = 1305 \text{ m}.$$

From the r_W(preclusion), r_{P_1} and d_{2W} values, we can compute r_{WP_1} and $r_{WP_1U_2}$ for both weapons.

$$r_{W_1P_1} = 1670 + 1000 = 2670 \text{ m}; \; r_{W_2P_2} = 2160 + 1000 = 3160 \text{ m as before}.$$

$$r_{W_1P_1U_2} = 2670 - 925 = 1745 \text{ m}; \; r_{W_2P_1U_2} = 3160 - 1355 = 1805 \text{ m}.$$

We also need to compute the constants relative to unit #1 and preclusion area #2. From the r_W(preclusion), r_{P_2} and d_{1W} values, we can compute $r_{WP_2U_1}$ for both weapons.

$$r_{W_1P_2U_1} = 2170 - 925 = 1245 \text{ m}; \; r_{W_2P_2U_1} = 2660 - 1355 = 1305 \text{ m}.$$

Thus $r_{WP_kU_{ij}} \equiv \max\{ r_{WP_kU_i}, r_{WP_kU_j} \}$ is easily calculated for $k = 1,2$ and $w = 1,2$:

$$r_{W_1P_1U_{12}} = 1745 \text{ m}; \; r_{W_2P_1U_{12}} = 1805 \text{ m}.$$

$$r_{W_1P_2U_{12}} = 1245 \text{ m}; \; r_{W_2P_1U_{12}} = 1305 \text{ m}.$$

Computing the distributions of the distance from preclusion areas 1 and 2

(1) *Unit #2:*

Preclusion area #1:

$$\mu_{x2P_1} \equiv (\mu_{x2} + \mu_{tx2}) - (x_{P_1} + \mu_{txP_1}) = (0 + 0) - (1000 + 0) = -1,000$$

$$\mu_{y2P_1} \equiv (\mu_{y2} + \mu_{ty2}) - (y_{P_1} + \mu_{tyP_1}) = (1100 + 0) - (1000 + 0) = 100$$

$$\sigma_{x2P_1}^2 \equiv (\sigma_{x2}^2 + \sigma_{tx2}^2) + (\sigma_{P_1}^2 + \sigma_{txP_1}^2) = (2000^2 + 75^2) + (0 + 0) = 4,005,625$$

$$\sigma_{y2P_1}^2 \equiv (\sigma_{y2}^2 + \sigma_{ty2}^2) + (\sigma_{P_1}^2 + \sigma_{tyP_1}^2) = (200^2 + 100^2) + (0 + 0) = 50,000$$

Let $\mu_{2P_1} \equiv \mu_{x2P_1}^2 + \mu_{y2P_1}^2 + \sigma_{x2P_1}^2 + \sigma_{y2P_1}^2$ and

$$\sigma_{2P_1}^2 \equiv 2\sigma_{x2P_1}^2(\sigma_{x2P_1}^2 + 2\mu_{x2P_1}^2) + 2\sigma_{y2P_1}^2(\sigma_{y2P_1}^2 + 2\mu_{y2P_1}^2)$$

Then

$$D_{2P_1}^2 \sim N[\mu_{2P_1}, \sigma_{2P_1}^2] \text{ or } D_{2P_1}^2 \sim N[5,065,625; 4.812 \times 10^{13}], \text{ with } \sigma_{2P_1} = 6,936,826$$

Thus

$$P[\ D_{2P_1}^2 > r_{W_1P_1}^2\ ] = P[\ Z > \frac{2670^2 - \mu_{2P_1}}{\sigma_{2P_1}}\ ] = P[\ Z > 0.297\ ] = 0.383$$

$$P[\ D_{2P_1}^2 > r_{W_2P_1}^2\ ] = P[\ Z > \frac{3160^2 - \mu_{2P_1}}{\sigma_{2P_1}}\ ] = P[\ Z > 0.709\ ] = 0.239$$

$$P[\ D_{2P_1}^2 > r_{W_1P_1U_2}^2\ ] = P[\ Z > \frac{1745^2 - \mu_{2P_1}}{\sigma_{2P_1}}\ ] = P[\ Z > -0.291\ ] = 0.615$$

$$P[\ D_{2P_1}^2 > r_{W_2P_1U_2}^2\ ] = P[\ Z > \frac{1805^2 - \mu_{2P_1}}{\sigma_{2P_1}}\ ] = P[\ Z > -0.261\ ] = 0.603$$

Preclusion area #2:

$$\mu_{x2P_2} \equiv (\ \mu_{x2} + \mu_{tx2}\ ) - (\ x_{P_2} + \mu_{txP_2}\ ) = (\ 0 + 0\ ) - (\ 500 + 0\ ) = -500$$

$$\mu_{y2P_2} \equiv (\ \mu_{y2} + \mu_{ty2}\ ) - (\ y_{P_2} + \mu_{tyP_2}\ ) = (\ 1100 + 0\ ) - (\ 1600 + 0\ ) = -500$$

$$\sigma_{x2P_2}^2 \equiv (\ \sigma_{x2}^2 + \sigma_{tx2}^2\ ) + (\ \sigma_{P_2}^2 + \sigma_{txP_2}^2\ ) = (\ 2000^2 + 75^2\ ) + (\ 0 + 0\ ) = 4,005,625$$

$$\sigma_{y2P_2}^2 \equiv (\ \sigma_{y2}^2 + \sigma_{ty2}^2\ ) + (\ \sigma_{P_2}^2 + \sigma_{tyP_2}^2\ ) = (\ 200^2 + 100^2\ ) + (\ 0 + 0\ ) = 50,000$$

Let $\mu_{2P_2} \equiv \mu_{x2P_2}^2 + \mu_{y2P_2}^2 + \sigma_{x2P_2}^2 + \sigma_{y2P_2}^2 = 4,555,625$

$$\sigma_{2P_2}^2 \equiv 2\sigma_{x2P_2}^2(\sigma_{x2P_2}^2 + 2\mu_{x2P_2}^2) + 2\sigma_{y2P_2}^2(\sigma_{y2P_2}^2 + 2\mu_{y2P_2}^2) = 3.615 \times 10^{13}$$

$$\sigma_{2P_2} = 6,012,544$$

Thus

$$P[\ D_{2P_2}^2 > r_{W_1P_2}^2\ ] = P[\ Z > \frac{2170^2 - \mu_{2P_2}}{\sigma_{2P_2}}\ ] = P[\ Z > 0.025\ ] = 0.490$$

$$P[\ D_{2P_2}^2 > r_{W_2P_2}^2\ ] = P[\ Z > \frac{2660^2 - \mu_{2P_2}}{\sigma_{2P_2}}\ ] = P[\ Z > 0.419\ ] = 0.338$$

$$P[\ D_{2P_2}^2 > r_{W_1P_2U_2}^2\ ] = P[\ Z > \frac{1245^2 - \mu_{2P_2}}{\sigma_{2P_2}}\ ] = P[\ Z > -0.500\ ] = 0.691$$

$$P[\ D_{2P_2}^2 > r_{W_2P_2U_2}^2\ ] = P[\ Z > \frac{1305^2 - \mu_{2P_2}}{\sigma_{2P_2}}\ ] = P[\ Z > -0.474\ ] = 0.682$$

Since $\overline{p}_{prec}(\ i\ |\ w,\ a_i,\ \overline{agg}_i\ ) = \min_k \left\{\ P[\ D_{iP_k}^2 > r_{WP_kU_i}^2\ ]\ \right\}$ and

$$P_{no\ shift}(\ i\ |\ w,\ a_i,\ \overline{agg}_i\ ) = \min_k \left\{\ P[\ D_{iP_k}^2 > r_{WP_k}^2\ ]\ \right\},$$

$$\overline{p}_{prec}(\ 2\ |\ w_1,\ a_2,\ \overline{agg}_2\ ) = \min(\ 0.615,\ 0.691\ ) = 0.615$$

$$\overline{p}_{prec}(\ 2\ |\ w_2,\ a_2,\ \overline{agg}_2\ ) = \min(\ 0.603,\ 0.682\ ) = 0.603$$

$$P_{no\ shift}(\ 2\ |\ w_1,\ a_2,\ \overline{agg}_2\ ) = \min(\ 0.383,\ 0.490\ ) = 0.383$$

$$P_{no\ shift}(\ 2\ |\ w_2,\ a_2,\ \overline{agg}_2\ ) = \min(\ 0.239,\ 0.338\ ) = 0.239$$

Since $p_{shift}(2 \mid w, a_2, a\overline{g}g_2) = \overline{p}_{prec}(2 \mid w, a_2, a\overline{g}g_2) - p_{no\ shift}(2 \mid w, a_2, a\overline{g}g_2)$,

$$p_{shift}(2 \mid w_1, a_2, a\overline{g}g_2) = 0.615 - 0.383 = 0.232$$

$$p_{shift}(2 \mid w_2, a_2, a\overline{g}g_2) = 0.603 - 0.239 = 0.364$$

NOTE: The closest preclusion area for target #2 is preclusion area #1, so all DGZ shifts will be computed from preclusion area #1.

(2) *Unit #1*:

Preclusion area #1: The distributions were calculated in section I.

Preclusion area #2:

$$\mu_{x1p_2} \equiv (\mu_{x1} + \mu_{tx1}) - (x_{p_2} + \mu_{txp_2}) = (0 + 0) - (500 + 0) = -500$$

$$\mu_{y1p_2} \equiv (\mu_{y1} + \mu_{ty1}) - (y_{p_2} + \mu_{typ_2}) = (0 + 0) - (1600 + 0) = -1600$$

$$\sigma^2_{x1p_2} \equiv (\sigma^2_{x1} + \sigma^2_{tx1}) + (\sigma^2_{p_2} + \sigma^2_{txp_2}) = (2000^2 + 75^2) + (0 + 0) = 4,005,625$$

$$\sigma^2_{y1p_2} \equiv (\sigma^2_{y1} + \sigma^2_{ty1}) + (\sigma^2_{p_2} + \sigma^2_{typ_2}) = (200^2 + 100^2) + (0 + 0) = 50,000$$

Let $\mu_{1p_2} \equiv \mu^2_{x1p_2} + \mu^2_{y1p_2} + \sigma^2_{x1p_2} + \sigma^2_{y1p_2} = 6,865,625$

$$\sigma^2_{1p_2} \equiv 2\sigma^2_{x1p_2}(\sigma^2_{x1p_2} + 2\mu^2_{x1p_2}) + 2\sigma^2_{y1p_2}(\sigma^2_{y1p_2} + 2\mu^2_{y1p_2}) = 3.661 \times 10^{13}$$

$$\sigma_{1p_2} = 6,050,842$$

Thus

$$P[D^2_{1p_2} > r^2_{W_1p_2}] = P\left[Z > \frac{2170^2 - \mu_{1p_2}}{\sigma_{1p_2}}\right] = P[Z > -0.356] = 0.639$$

$$P[D^2_{1p_2} > r^2_{W_2p_2}] = P\left[Z > \frac{2660^2 - \mu_{1p_2}}{\sigma_{1p_2}}\right] = P[Z > 0.035] = 0.486$$

$$P[D^2_{1p_2} > r^2_{W_1p_2U_1}] = P\left[Z > \frac{1245^2 - \mu_{1p_2}}{\sigma_{1p_2}}\right] = P[Z > -0.878] = 0.810$$

$$P[D^2_{1p_2} > r^2_{W_2p_2U_1}] = P\left[Z > \frac{1305^2 - \mu_{1p_2}}{\sigma_{1p_2}}\right] = P[Z > -0.853] = 0.803$$

Since $\overline{p}_{prec}(i \mid w, a_i, a\overline{g}g_i) = \min_k \left\{ P[D^2_{ip_k} > r^2_{WP_kU_i}] \right\}$ and

$$p_{no\ shift}(i \mid w, a_i, a\overline{g}g_i) = \min_k \left\{ P[D^2_{ip_k} > r^2_{WP_k}] \right\},$$

$$\overline{p}_{prec}(1 \mid w_1, a_1, a\overline{g}g_1) = \min(0.668, 0.810) = 0.668$$

$$\overline{p}_{prec}(1 \mid w_2, a_1, a\overline{g}g_1) = \min(0.656, 0.803) = 0.656$$

$$p_{no\ shift}(1 \mid w_1, a_1, a\overline{g}g_1) = \min(0.439, 0.639) = 0.439$$

$$p_{no\ shift}(1 \mid w_2, a_1, a\overline{g}g_1) = \min(0.286, 0.486) = 0.286$$

Since $p_{shift}(1 | w, a_1, a\overline{g}g_1) = \overline{p}_{prec}(1 | w, a_1, a\overline{g}g_1) - p_{no\ shift}(1 | w, a_1, a\overline{g}g_1)$,

$p_{shift}(1 | w_1, a_1, a\overline{g}g_1) = 0.668 - 0.439 = 0.229$

$p_{shift}(1 | w_2, a_1, a\overline{g}g_1) = 0.656 - 0.286 = 0.370$

NOTE: The closest preclusion area for target unit #1 is preclusion area #1, so all DGZ shifts will be computed from preclusion area #1. As a result, the probabilities of $\overline{p}_{prec}(1 | w, a_1, a\overline{g}g_1)$, $p_{no\ shift}(1 | w, a_1, a\overline{g}g_1)$, and $p_{shift}(1 | w, a_1, a\overline{g}g_1)$ remain the same as calculated in section #1.

Computing the distributions of the distance between target units 1 and 2

The differences between the perceived locations of targets 1 and 2 are:

X-coordinate: $(X_1 + TX_1) - (X_2 + TX_2)$

Y-coordinate: $(Y_1 + TY_1) - (Y_2 + TY_2)$

Thus we define

$\mu_{x12} \equiv (\mu_{x1} + \mu_{tx1}) - (\mu_{x2} + \mu_{tx2}) = (0 + 0) - (0 + 0) = 0$

$\mu_{y12} \equiv (\mu_{y1} + \mu_{ty1}) - (\mu_{y2} + \mu_{ty2}) = (0 + 0) - (1100 + 0) = 1100$

$\sigma^2_{x12} \equiv (\sigma^2_{x1} + \sigma^2_{tx1}) + (\sigma^2_{x2} + \sigma^2_{tx2}) + 2(-1)\rho_{x12}\sigma_{x1}\sigma_{x2}$

$\qquad = (2000^2 + 75^2) + (2000^2 + 75^2) - 2(0.5)(2000)(2000) = 4,011,250$

$\sigma^2_{y12} \equiv (\sigma^2_{y1} + \sigma^2_{ty1}) + (\sigma^2_{y2} + \sigma^2_{ty2}) + 2(-1)\rho_{y12}\sigma_{y1}\sigma_{y2}$

$\qquad = (200^2 + 100^2) + (200^2 + 100^2) - 2(0.9)(200)(200) = 28,000$

Let $\mu_{12} \equiv \mu^2_{x12} + \mu^2_{y12} + \sigma^2_{x12} + \sigma^2_{y12} = 9,249,250$

$\sigma^2_{12} \equiv 2\sigma^2_{x12}(\sigma^2_{x12}+2\mu^2_{x12}) + 2\sigma^2_{y12}(\sigma^2_{y12}+2\mu^2_{y12}) = 1.285 \times 10^{14}$

$\sigma_{12} = 11,335,667$

Then

$$P[D^2_{12} > (d_{1W_1} + d_{2W_1})^2] = P[Z > \frac{(2\cdot925)^2 - \mu_{12}}{\sigma_{12}}] = P[Z > -0.051] = 0.520$$

$$P[D^2_{12} > (d_{1W_2} + d_{2W_2})^2] = P[Z > \frac{(2\cdot1355)^2 - \mu_{12}}{\sigma_{12}}] = P[Z > -0.295]$$
$$= 0.384$$

Thus $p_{aggr}(12 | w_1, a_{12}) = 1 - 0.520 = 0.480$ and $p_{aggr}(12 | w_2, a_{12}) = 1 - 0.384 = 0.616$.

Computing the aggregation and engagement probabilities

(1) *Unit #1:*

Recall that $\overline{p}_{aggr}(i \mid w, a_i) = 1 - \max_j \left\{ p_{aggr}(ij \mid w, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j \mid w, a_j, a\overline{g}g_j) \right\}$. In this case, for $i = 1$, there is only one $j \neq i : j = 2$. Previously we found the following:

$\overline{p}_{prec}(2 \mid w_1, a_2, a\overline{g}g_2) = 0.615$ and $\overline{p}_{prec}(2 \mid w_2, a_2, a\overline{g}g_2) = 0.603$.

From the previous paragraph, we have:

$p_{aggr}(12 \mid w_1, a_{12}) = 0.480$ and $p_{aggr}(12 \mid w_2, a_{12}) = 0.616$.

If we continue to assume that $p_{avail}(1) = p_{avail}(2) = 1.0$,

$\overline{p}_{aggr}(1 \mid w_1, a_1) = 1 - \left\{ 0.480 \cdot 1.0 \cdot 0.615 \right\} = 0.705$, and

$\overline{p}_{aggr}(1 \mid w_2, a_1) = 1 - \left\{ 0.616 \cdot 1.0 \cdot 0.603 \right\} = 0.629$.

(2) *Unit #2:*

For $i = 2$, there is only one $j \neq i : j = 1$. In section I, we found the following:

$\overline{p}_{prec}(1 \mid w_1, a_1, a\overline{g}g_1) = 0.668$ and $\overline{p}_{prec}(1 \mid w_2, a_1, a\overline{g}g_1) = 0.656$.

From the previous paragraph, we have:

$p_{aggr}(12 \mid w_1, a_{12}) = 0.480$ and $p_{aggr}(12 \mid w_2, a_{12}) = 0.616$.

If we continue to assume that $p_{avail}(1) = p_{avail}(2) = 1.0$,

$\overline{p}_{aggr}(2 \mid w_1, a_2) = 1 - \left\{ 0.480 \cdot 1.0 \cdot 0.668 \right\} = 0.679$, and

$\overline{p}_{aggr}(2 \mid w_2, a_2) = 1 - \left\{ 0.616 \cdot 1.0 \cdot 0.656 \right\} = 0.596$.

From this, we can compute conditional probabilities of engagement:

$p_{engage}(i, \overline{s} \mid w) = p_{avail}(i) \cdot p_{no\ shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i)$,

$p_{engage}(i, s \mid w) = p_{avail}(i) \cdot p_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i)$,

$p_{engage}(i, \overline{s}, w) = p_{engage}(i, \overline{s} \mid w) \cdot p_{round}(i, w)$, and

$p_{engage}(i, s, w) = p_{engage}(i, s \mid w) \cdot p_{round}(i, w)$.

In the previous section, we assumed that $p_{round}(1, w_1) = 0.75$ and $p_{round}(1, w_2) = 0.25$. Let us assume that $p_{round}(2, w_1) = 0.25$ and $p_{round}(2, w_2) = 0.75$.

(3) *Unit #1:*

   (a) *Weapon #1:*

$p_{engage}(1, \overline{s} \mid w_1) = 1.0 \cdot 0.439 \cdot 0.705 = 0.309$

$p_{engage}(1, s \mid w_1) = 1.0 \cdot 0.229 \cdot 0.705 = 0.161$

$p_{engage}(1, \overline{s}, w_1) = 0.309 \cdot 0.75 \cdot 1.0 = 0.232$

$p_{engage}(1, s, w_1) = 0.161 \cdot 0.75 \cdot 1.0 = 0.121$

   (b) *Weapon #2:*

$p_{engage}(1, \overline{s} \mid w_2) = 1.0 \cdot 0.286 \cdot 0.629 = 0.180$

$p_{engage}(1, s \mid w_2) = 1.0 \cdot 0.370 \cdot 0.629 = 0.233$

$p_{engage}(1, \overline{s}, w_2) = 0.180 \cdot 0.25 \cdot 1.0 = 0.045$

$p_{engage}(1, s, w_2) = 0.233 \cdot 0.25 \cdot 1.0 = 0.058$

(4) *Unit #2:*

   (a) *Weapon #1:*

$p_{engage}(2, \overline{s} \mid w_1) = 1.0 \cdot 0.383 \cdot 0.679 = 0.260$

$p_{engage}(2, s \mid w_1) = 1.0 \cdot 0.232 \cdot 0.679 = 0.158$

$p_{engage}(2, \overline{s}, w_1) = 0.260 \cdot 0.25 \cdot 1.0 = 0.065$

$p_{engage}(2, s, w_1) = 0.158 \cdot 0.25 \cdot 1.0 = 0.040$

(b) *Weapon #2:*

$p_{engage}(2, \overline{s} \mid w_2) = 1.0 \cdot 0.239 \cdot 0.596 = 0.142$

$p_{engage}(2, s \mid w_2) = 1.0 \cdot 0.364 \cdot 0.596 = 0.217$

$p_{engage}(2, \overline{s}, w_2) = 0.142 \cdot 0.75 \cdot 1.0 = 0.107$

$p_{engage}(2, s, w_2) = 0.217 \cdot 0.75 \cdot 1.0 = 0.163$

Computing the AGZ and conditional defeat distributions of Unit #2 engaged as a single target

(1) *No DGZ shift, Weapon type 1:*

$X_{AGZ_1} = X_2 + TX_2 + CX_2 \; ; \; Y_{AGZ_1} = Y_2 + TY_2 + CY_2.$ Thus

$X_{AGZ_1} - X_2 = TX_2 + CX_2 \; ; \; Y_{AGZ_1} - Y_2 = TY_2 + CY_2.$

Let $\mu_{x2AGZ_1} \equiv E[X_{AGZ_1}-X_2] = \mu_{tx2} + \mu_{cx2}$; $\mu_{y2AGZ_1} \equiv E[Y_{AGZ_1}-Y_2] = \mu_{ty2} + \mu_{cy2}$.

$\sigma^2_{x2AGZ_1} \equiv \text{Var}[X_{AGZ_1}-X_2] = \sigma^2_{tx2} + \sigma^2_{cx2}$; $\sigma^2_{y2AGZ_1} \equiv \text{Var}[Y_{AGZ_1}-Y_2] = \sigma^2_{ty2} + \sigma^2_{cy2}$.

$\mu_{2AGZ_1} \equiv \mu^2_{x2AGZ_1} + \mu^2_{y2AGZ_1} + \sigma^2_{x2AGZ_1} + \sigma^2_{y2AGZ_1}$

$\sigma^2_{2AGZ_1} \equiv 2\sigma^2_{x2AGZ_1}(\sigma^2_{x2AGZ_1}+2\mu^2_{x2AGZ_1}) + 2\sigma^2_{y2AGZ_1}(\sigma^2_{y2AGZ_1}+2\mu^2_{y2AGZ_1})$

Then the squared distance between the AGZ for weapon of type #1 and the unit 2, $D^2_{2AGZ_1}$, is:

$$D^2_{2AGZ_1} = D^2_{x2AGZ_1} + D^2_{y2AGZ_1} \sim N[\mu_{2AGZ_1}, \sigma^2_{2AGZ_1}]$$

Evaluating the terms,

$\mu_{x2AGZ_1} = 0$; $\mu_{y2AGZ_1} = 0$

$\sigma^2_{x2AGZ_1} = 75^2 + 225^2 = 56,250$; $\sigma^2_{y2AGZ_1} = 100^2 + 225^2 = 60,625$.

$\mu_{2AGZ_1} = 0 + 0 + 56,250 + 60,625 = 116,875$

$\sigma^2_{2AGZ_1} = 2(56,250)^2 + 2(60,625)^2 = 13,678,906,250$

The conditional probability of defeat given weapon type 1 and no DGZ shift is:

$$P_{defeat}(2 \mid w_1, a_1, \overline{s}, a\overline{g}g_1) = P[D^2_{1AGZ_1} \leq d^2_{1W_1}] = P[Z \leq \frac{925^2 - \mu_{1AGZ_1}}{\sigma_{1AGZ_1}}] = P[Z \leq 6.32]$$
$$\doteq 1.0$$

We note that this result is identical to the result obtained against unit #1 with weapon #1, no DGZ shift; this is due to the fact that the units are identical.

(2) *No DGZ shift, Weapon type 2:*

Again, the units are identical, so the result is identical to the result obtained against unit #1 with weapon #2, no DGZ shift:

$$P_{defeat}(2 \mid w_2, a_1, \overline{s}, a\overline{g}g_1) = P[D^2_{1AGZ_2} \leq d^2_{1W_2}] = P[Z \leq \frac{1355^2 - \mu_{1AGZ_2}}{\sigma_{1AGZ_2}}]$$
$$= P[Z \leq 29.209] \doteq 1.0$$

(3) *DGZ shift. Weapon type 1:*

Recall that $r_{W_1P_1} = 2670$ and $E[D^2_{2P_1}] = 5,065,625$. Thus

$$\beta_1 = \frac{r_{W_1P_1}}{\sqrt{E[D^2_{2P_1}]}} = 1.186$$

The shifted DGZ coordinates are:

$$X_{AGZ^s_1} = \beta X_{2L} + (1-\beta)x_{P_1} = \beta(X_2+TX_2) + (1-\beta)x_{P_1}$$
$$Y_{AGZ^s_1} = \beta Y_{2L} + (1-\beta)y_{P_1} = \beta(Y_2+TY_2) + (1-\beta)y_{P_1}$$

Thus

$$X_{AGZ_1^s} - X_2 = (\beta - 1)X_2 + \beta TX_2 + (1 - \beta)x_{P_1} + CX_2 \text{ and}$$

$$Y_{AGZ_1^s} - Y_2 = (\beta - 1)Y_2 + \beta TY_2 + (1 - \beta)y_{P_1} + CY_2.$$

Let $\mu_{x2AGZ_1^s} \equiv E[X_{AGZ_1^s} - X_2] = (\beta - 1)\mu_{x2} + \beta\mu_{tx2} + (1 - \beta)x_{P_1} + \mu_{cx2}$

$\mu_{y2AGZ_1^s} \equiv E[Y_{AGZ_1^s} - Y_2] = (\beta - 1)\mu_{y2} + \beta\mu_{ty2} + (1 - \beta)y_{P_1} + \mu_{cy2}$

$\sigma^2_{x2AGZ_1^s} \equiv \text{Var}[X_{AGZ_1^s} - X_2] = (\beta - 1)^2\sigma^2_{x2} + \beta^2\sigma^2_{tx2} + \sigma^2_{cx2}$

$\sigma^2_{y2AGZ_1^s} \equiv \text{Var}[Y_{AGZ_1^s} - Y_2] = (\beta - 1)^2\sigma^2_{y2} + \beta^2\sigma^2_{ty2} + \sigma^2_{cy2}$

$\mu_{2AGZ_1^s} \equiv \mu^2_{x2AGZ_1^s} + \mu^2_{y2AGZ_1^s} + \sigma^2_{x2AGZ_1^s} + \sigma^2_{y2AGZ_1^s}$

$\sigma^2_{2AGZ_1^s} \equiv 2\sigma^2_{x2AGZ_1^s}(\sigma^2_{x2AGZ_1^s} + 2\mu^2_{x2AGZ_1^s}) + 2\sigma^2_{y2AGZ_1^s}(\sigma^2_{y2AGZ_1^s} + 2\mu^2_{y2AGZ_1^s})$

Then the squared distance between the AGZ (based on a shifted DGZ) for weapon of type #1 and the unit 2, $D^2_{2AGZ_1}$, is:

$$D^2_{2AGZ_1^s} = D^2_{x2AGZ_1^s} + D^2_{y2AGZ_1^s} \stackrel{.}{\sim} N[\mu_{2AGZ_1^s}, \sigma^2_{2AGZ_1^s}]$$

Evaluating the terms,

$\mu_{x2AGZ_1^s} = 0.085(0) + 1.085(0) - 0.085(1000) + 0 = -85 = \mu_{y2AGZ_1^s}$

$\sigma^2_{x2AGZ_1^s} = (0.085)^2(2000^2) + (1.085)^2(75^2) + 225^2 = 85,628$

$\sigma^2_{y2AGZ_1^s} = (0.085)^2(200^2) + (1.085)^2(100^2) + 225^2 = 61,764$

$\mu_{2AGZ_1^s} = 2(-85)^2 + 85,628 + 61,764 = 161,842$

$\sigma^2_{2AGZ_1^s} = 2(85,628)(85,628 + (2)(-85)^2) + 2(61,764)(61,764 + (2)(-85)^2) = 26,553,567,387$

$\sigma_{2AGZ_1^s} = 162,953$

The conditional probability of defeat given weapon type #1 and a DGZ shift is:

$$P_{defeat}(2 \mid w_1, a_2, s, a\overline{g}g_2) = P[D^2_{2AGZ_1^s} \leq d^2_{2W_1}] = P[Z \leq \frac{925^2 - \mu_{2AGZ_1^s}}{\sigma_{2AGZ_1^s}}]$$

$$= P[Z \leq 4.258] \doteq 1.0$$

(4) *DGZ shift, Weapon type 2:*

Recall that $r_{W_2P_1} = 3160$ and $E[D^2_{2P_1}] = 5,065,625$. Thus

$$\beta_2 = \frac{r_{W_2P_1}}{\sqrt{E[D^2_{2P_1}]}} = 1.404$$

The squared distance between the AGZ (based on a shifted DGZ) for weapon of type #2 and the unit #2, $D^2_{2AGZ_2}$, is evaluated as before:

$$D^2_{2AGZ^s_2} = D^2_{x2AGZ^s_2} + D^2_{y2AGZ^s_2} \sim N[\mu_{2AGZ^s_2}, \sigma^2_{2AGZ^s_2}]$$

Evaluating the terms,

$$\mu_{x2AGZ^s_2} = -0.284(1000) = -284 = \mu_{y2AGZ^s_2}$$
$$\sigma^2_{x2AGZ^s_2} = (0.284)^2(2000^2) + (1.284)^2(75^2) + 150^2 = 352{,}346$$
$$\sigma^2_{y2AGZ^s_2} = (0.284)^2(200^2) + (1.284)^2(100^2) + 150^2 = 42{,}213$$
$$\mu_{2AGZ^s_2} = 555{,}871$$
$$\sigma^2_{2AGZ^s_2} = 379{,}154{,}252{,}695; \quad \sigma_{2AGZ^s_2} = 615{,}755$$

The conditional probability of defeat given weapon type 2 and a DGZ shift is:

$$P_{defeat}(2\,|\,w_2, a_2, s, a\overline{gg}_2) = P[\,D^2_{2AGZ^s_2} \le d^2_{2\,W_2}\,] = P[\,Z \le \frac{1355^2 - \mu_{2AGZ^s_2}}{\sigma_{2AGZ^s_2}}\,]$$
$$= P[\,Z \le 2.079\,] = 0.981$$

Section IV - COMPUTING THE DGZ AND DISTRIBUTIONS OF THE AGGREGATE TARGET

Computing α

Assume that target unit #1 has priority 1 and target unit #2 has priority 2. Recall that $r_{T_1} = r_{T_2} = 500$ m. To compute α, we need to solve for $\hat{D}_{12} \equiv \sqrt{E[\,D^2_{12}\,|\,0 \le D^2_{12} \le (\,d_{1\,W} + d_{2\,W})^2\,]}$ for $w = 1,2$. From p. 33 of CAA-RP-89-3, if a variable $U \sim N(\,\mu, \sigma^2\,)$ then

$$E[\,U\,|\,A \le U \le B\,] = \mu + \frac{Z\left[\frac{A-\mu}{\sigma}\right] - Z\left[\frac{B-\mu}{\sigma}\right]}{\Phi\left[\frac{B-\mu}{\sigma}\right] - \Phi\left[\frac{A-\mu}{\sigma}\right]}\,\sigma\,,$$

where $Z(u) = \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}$ and $\Phi(u)$ is the standard normal integral evaluated at u.

In our problem, $U = D^2_{12}$, $B = (\,d_{1\,W} + d_{2\,W})^2$, $A = 0$, $\mu = E[\,D^2_{12}\,]$ and $\sigma = \sqrt{Var[\,D^2_{12}\,]}$.

Weapon #1:

Recall that $d_{1\,W_1} = d_{2\,W_1} = 925$, and $E[\,D_{12}^2\,] = 9,249,250$. Then using the priority and target unit size criteria for establishing α. In this problem, $U = D_{12}^2$, $B = (\,d_{1\,W_1} + d_{2\,W_1}\,)^2$, $A = 0$, $\mu = E[\,D_{12}^2\,]$ and $\sigma = \sqrt{\,Var[\,D_{12}^2\,]\,}$. Thus

$$\hat{D}_{12}^2 = E[\,D_{12}^2 \mid D_{12}^2 \le (\,d_{1\,W_1} + d_{2\,W_1}\,)^2\,] = 9,249,250 + \left[\frac{.286 - .350}{.304 - .207}\right](\,11,335,667\,) = 1,768,300$$

and $\hat{D}_{12} = 1330$.

$$\alpha_1 = \min\left\{ \frac{d_{2\,W_1}}{\hat{D}_{12}} \; ; \right.$$

$$\left. \frac{\hat{D}_{12} - d_{1\,W_1}}{\hat{D}_{12}} + \frac{r_{T_1}}{r_{T_1} + r_{T_2}}\left[\frac{d_{2\,W_1}}{\hat{D}_{12}} - \frac{\hat{D}_{12} - d_{1\,W_1}}{\hat{D}_{12}} \right] \cdot \left[1 + \frac{[2] - [1]}{\max\{\,[1],\,[2]\,\}} \right] \right\}$$

Thus

$$\alpha_1 = \min\left\{ 0.695;\; 0.305 + \tfrac{1}{2}\,[\,0.695 - 0.305\,]\,[\,1 + \tfrac{1}{2}\,] \right\} = \min\{\,0.695;\; 0.598\,\} = 0.598$$

Thus the aggregate target DGZ_1^a for weapon #1 has the following coordinates:

$$X_{DGZ_1^a} = \alpha_1 X_{1\,L} + (\,1 - \alpha_1\,) X_{2\,L};\; \mu_{x\,DGZ_1^a} = 0.598(0) + (1 - 0.598)(0) = 0$$

$$Y_{DGZ_1^a} = \alpha_1 Y_{1\,L} + (\,1 - \alpha_1\,) Y_{2\,L};\; \mu_{y\,DGZ_1^a} = 0.598(0) + (1 - 0.598)(1100) = 442.2 \doteq 442$$

(1) *Computing the distributions between DGZ_1^a and preclusion areas 1 and 2:*

(a) *Preclusion area #1:*

We can adjust for the fact that the target units are aggregated, thus $D_{12}^2 \le d_{12\,W_1}^2$.

$$\bar{P}_{prec}(\,ij \mid w_1, a_{ij}, agg_{ij}\,) = \min_k\left\{ P[\,D_{D\,P_k}^2 > r_{WP_k\,U_{ij}}^2 \mid D_{12}^2 \le d_{12\,W_1}^2\,] \right\} \text{ and}$$

$$P_{no\,shift}(\,ij \mid w_1, a_{ij}, agg_{ij}\,) = \min_k\left\{ P[\,D_{D\,P_k}^2 > r_{WP_k}^2 \mid D_{12}^2 \le d_{12\,W_1}^2\,] \right\}.$$

Recall that $E[D_{12}^2] = 9,249,250 \qquad Var[D_{12}^2] = 11,335,667^2$

$$E[D_{1\,P_1}^2] = 6,055,625 \qquad Var[D_{1\,P_1}^2] = 6,951,084^2$$

$$E[D_{2\,P_1}^2] = 5,065,625 \qquad Var[D_{2\,P_1}^2] = 6,936,826^2$$

Then using the formula on page 46 with $A=0$, $B = d_{12\,W_1}^2 = (2 \cdot 925)^2$, $\mu = E[D_{12}^2]$ and $\sigma^2 = Var[D_{12}^2]$,

$$E[D_{12}^2 \mid D_{12}^2 \le d_{12\,W_1}^2] = 9,249,250 + \left[\frac{.286 - .350}{.304 - .207}\right](\,11,335,667\,) = 1,768,300$$

$$\text{Var}[\,D^2_{12}\,|\,D^2_{12} \le d^2_{12\,W_1}\,] = \left\{\; 1 + \left[\frac{(-.816)(.286)-(-.514)(.350)}{.304-.207}\right] - \left[\frac{.286-.350}{.304-.207}\right]^2 \;\right\} \cdot 11,335,667$$

$$= 0.014 \cdot 11,335,667^2 = 1.799 \text{ x } 10^{12} = 1,341,270^2$$

Thus for $\alpha_1 = 0.598$,

$$\text{E}[\,D^2_{D\,P_1}\,|\,D^2_{12} \le d^2_{12\,W_1}\,] = (\alpha_1{}^2-\alpha_1)\,\text{E}[\,D^2_{12}\,|\,D^2_{12} \le d^2_{12\,W_1}\,] + \alpha_1\,\text{E}[\,D^2_{1\,P_1}\,] + (1-\alpha_1)\,\text{E}[\,D^2_{2\,P_1}\,]$$

$$= (-0.240)(1,768,300) + (0.598)(6,055,625) + (0.402)(5,065,625) = 5,232,553$$

$$\text{Var}[\,D^2_{D\,P_1}\,|\,D^2_{12} \le d^2_{12\,W_1}\,]$$

$$= (\alpha_1{}^2-\alpha_1)^2\,\text{Var}[\,D^2_{12}\,|\,D^2_{12}\le d^2_{12\,W_1}\,] + \alpha_1{}^2\,\text{Var}[\,D^2_{1\,P_1}\,] + (1-\alpha_1)^2\,\text{Var}[\,D^2_{2\,P_1}\,]$$

$$= (-0.240)^2(1,341,270)^2 + (0.598)^2(6,951,084)^2 + (0.402)^2(6,936,826)^2$$

$$= 2.516 \text{ x } 10^{13}; \text{ with } \sqrt{\text{Var}[\,D^2_{D\,P_1}\,|\,D^2_{12} \le d^2_{12\,W_1}\,]} = 5,051,820$$

Thus

$$P[\,D^2_{D\,P_1} > r^2_{W_1P_1U_{12}}\,|\,D^2_{12} \le d^2_{12\,W_1}\,] = P[\,Z > \frac{1745^2 - 5,232,553}{5,051,820}\,] = P[\,Z > -0.436\,] = 0.669$$

$$P[\,D^2_{D\,P_1} > r^2_{W_1P_1}\,|\,D^2_{12} \le d^2_{12\,W_1}\,] = P[\,Z > \frac{2670^2 - 5,232,553}{5,051,820}\,] = P[\,Z > 0.378\,] = 0.353$$

(b) Preclusion area #2:

Recall that

$$\text{E}[\,D^2_{12}\,|\,D^2_{12} \le d^2_{12\,W_1}\,] = 1,768,300 \qquad \text{Var}[\,D^2_{12}\,|\,D^2_{12} \le d^2_{12\,W_1}\,] = 1,341,270^2$$

$$\text{E}[\,D^2_{1\,P_2}\,] = 6,865,625 \qquad \text{Var}[\,D^2_{1\,P_2}\,] = 6,050,842^2$$

$$\text{E}[\,D^2_{2\,P_2}\,] = 4,555,625 \qquad \text{Var}[\,D^2_{2\,P_2}\,] = 6,012,544^2$$

Thus for $\alpha_1 = 0.598$,

$$\text{E}[\,D^2_{D\,P_2}\,|\,D^2_{12} \le d^2_{12\,W_1}\,] = (\alpha_1{}^2-\alpha_1)\,\text{E}[\,D^2_{12}\,|\,D^2_{12} \le d^2_{12\,W_1}\,] + \alpha_1\,\text{E}[\,D^2_{1\,P_2}\,] + (1-\alpha_1)\,\text{E}[\,D^2_{2\,P_2}\,]$$

$$= (-0.240)(1,768,300) + (0.598)(6,865,625) + (0.402)(4,555,625) = 5,512,613$$

$$\text{Var}[\,D^2_{D\,P_2}\,|\,D^2_{12} \le d^2_{12\,W_1}\,]$$

$$= (\alpha_1{}^2-\alpha_1)^2\,\text{Var}[\,D^2_{12}\,|\,D^2_{12}\le d^2_{12\,W_1}\,] + \alpha_1{}^2\,\text{Var}[\,D^2_{1\,P_2}\,] + (1-\alpha_1)^2\,\text{Var}[\,D^2_{2\,P_2}\,]$$

$$= (-0.240)^2(1,341,270)^2 + (0.598)^2(6,050,842)^2 + (0.402)^2(6,012,544)^2 =$$

$$= 1.904 \text{ x } 10^{13}; \text{ with } \sqrt{\text{Var}[\,D^2_{D\,P_2}\,|\,D^2_{12} \le d^2_{12\,W_1}\,]} = 4,363,360$$

Thus

$$P[\, D^2_{D\,P_2} > r^2_{W_1 P_2 U_{12}} \,|\, D^2_{12} \le d^2_{12\,W_1} \,] = P[\, Z > \frac{1245^2 - 5{,}512{,}613}{4{,}363{,}360} \,] = P[\, Z > -0.908 \,] = 0.818$$

$$P[\, D^2_{D\,P_2} > r^2_{W_1 P_2} \,|\, D^2_{12} \le d^2_{12\,W_1} \,] = P[\, Z > \frac{2170^2 - 5{,}512{,}613}{4{,}363{,}360} \,] = P[\, Z > -0.184 \,] = 0.573$$

Thus for weapon #1,

$$
\begin{aligned}
\overline{P}_{prec}(\,12\,|\,w_1, a_{12}, agg_{12}\,) \quad &= \min_k \left\{\, P[\, D^2_{D\,P_k} > r^2_{W_1 P_k U_{12}} \,|\, D^2_{12} \le d^2_{12\,W_1} \,] \,\right\} \\
&= \min \left\{\, 0.669,\ 0.818 \,\right\} \\
&= 0.669
\end{aligned}
$$

$$
\begin{aligned}
P_{no\ shift}(\,12\,|\,w_1, a_{12}, agg_{12}\,) \quad &= \min_k \left\{\, P[\, D^2_{D\,P_k} > r^2_{W_1 P_k} \,|\, D^2_{12} \le d^2_{12\,W_1} \,] \,\right\}. \\
&= \min \left\{\, 0.353,\ 0.573 \,\right\} \\
&= 0.353
\end{aligned}
$$

NOTE: The closest preclusion area for the aggregate target formed from units #1 and #2, weapon #1 is preclusion area #1, so all weapon #1 DGZ shifts will be computed from preclusion area #1.

(2) *Computing the probability of a DGZ_1 shift:*

To evaluate the probability of a shift in the DGZ for the aggregate target formed from target units #1 and #2, engaged with weapon type #1, we need to calculate the distribution of the shifted DGZ.

We begin by evaluating β_1. To do this, we need to evaluate

$$E[\, D^2_{D\,P_1} \,|\, D^2_{12} \le (d_1{}_{W_1} + d_2{}_{W_1})^2 \cap \alpha_1 r^2_{W_1 P_1 U_1} + (1 - \alpha_1) r^2_{W_1 P_1 U_2}$$

$$\le \alpha_1 D^2_{1\,P_1} + (1 - \alpha_1) D^2_{2\,P_1} < r^2_{W_1 P_1} + (\alpha_1 - \alpha_1^2)(d_1{}_{W_1} + d_2{}_{W_1})^2 \,]$$

$$\doteq (\alpha_1^2 - \alpha_1)\, E[\, D^2_{12} \,|\, D^2_{12} \le (d_1{}_{W_1} + d_2{}_{W_1})^2 \,] + E[\![\, \alpha_1 D^2_{1\,P_1} + (1 - \alpha_1) D^2_{2\,P_1} \,|\, \alpha_1 r^2_{W_1 P_1 U_1}$$

$$+ (1 - \alpha_1) r^2_{W_1 P_1 U_2} \le \alpha_1 D^2_{1\,P_1} + (1 - \alpha_1) D^2_{2\,P_1} < r^2_{W_1 P_1} + (\alpha_1 - \alpha_1^2)(d_1{}_{W_1} + d_2{}_{W_1})^2 \,]\!]$$

$E[\, D^2_{12} \,|\, D^2_{12} \le (d_1{}_{W_1} + d_2{}_{W_1})^2 \,]$ has been evaluated previously as 1,768,300.

To evaluate $E[\![\, \alpha_1 D^2_{1\,P_1} + (1 - \alpha_1) D^2_{2\,P_1} \,|\, \alpha_1 r^2_{W_1 P_1 U_1} + (1 - \alpha_1) r^2_{W_1 P_1 U_2} \le \alpha_1 D^2_{1\,P_1} +$ $(1 - \alpha_1) D^2_{2\,P_1} < r^2_{W_1 P_1} + (\alpha_1 - \alpha_1^2)(d_1{}_{W_1} + d_2{}_{W_1})^2 \,]\!]$, we need to determine the conditional distribution of $\alpha_1 D^2_{1\,P_1} + (1 - \alpha_1) D^2_{2\,P_1}$.

Recall that

$$\mu_{iP_I} \equiv \mu^2_{xiP_I} + \mu^2_{yiP_I} + \sigma^2_{xiP_I} + \sigma^2_{yiP_I}, \text{ and}$$

$$\sigma^2_{iP_I} \equiv 2\sigma^2_{xiP_I}(\sigma^2_{xiP_I} + 2\mu^2_{xiP_I}) + 2\sigma^2_{yiP_I}(\sigma^2_{yiP_I} + 2\mu^2_{yiP_I}).$$

Then $D^2_{iP_I} \sim N[\mu_{iP_I}, \sigma^2_{iP_I}]$ and the conditional expectation given $\alpha_I r^2_{W_I P_I U_i} + (1-\alpha_I) r^2_{W_I P_I U_j}$

$\leq \alpha_I D^2_{iP_I} + (1-\alpha_I) D^2_{jP_I} < r^2_{W_I P_I} + (\alpha_I - \alpha^2_I)(d_{IW_I} + d_{2W_I})^2$ can be found in the usual manner.

$$\mu_{IP_I} = 6,055,625 \qquad\qquad \mu_{2P_I} = 5,065,625$$

$$\sigma_{IP_I} = 6,951,084 \qquad\qquad \sigma_{2P_I} = 6,936,827$$

Then $\alpha_I \mu_{IP_I} + (1-\alpha_I)\mu_{2P_I} = 5,657,645$

$$\alpha_I^2 \sigma^2_{IP_I} + (1-\alpha_I)^2 \sigma^2_{2P_I} = 2.505 \times 10^{13}; \text{ square root} = 5,005,484.$$

Let $U \equiv \alpha_I D^2_{IP_I} + (1-\alpha_I)D^2_{2P_I}$; $\mu_U = 5,657,645$; $\sigma_U = 5,005,484$;

$$A \equiv \alpha_I r^2_{W_I P_I U_I} + (1-\alpha_I) r^2_{W_I P_I U_2}; \quad B \equiv r^2_{W_I P_I} + (\alpha_I - \alpha^2_I)(d_{IW_I} + d_{2W_I})^2; \text{ then}$$

Then $E[\, U \mid A \leq U < B\,] = 5,657,645 + \left[\dfrac{.355 - .353}{.691 - .314}\right](5,005,484) = 5,684,199$

Thus $E[\, D^2_{DP_I} \mid D^2_{I2} \leq (d_{IW_I} + d_{2W_I})^2 \cap A \leq U < B\,]$

$$\doteq (0.598^2 - 0.598)(1,768,300) + 5,684,199 = 5,259,107.$$

Recalling that $r_{W_I P_I} = 2670$, and defining

$$\hat{D}_{DP_I} \equiv \sqrt{E[\, D^2_{DP_I} \mid D^2_{I2} \leq (d_{IW_I} + d_{2W_I})^2 \cap A \leq U < B\,]}, \text{ then } \hat{\beta}_I = \dfrac{r_{W_I P_I}}{\hat{D}_{DP_I}} = 1.164$$

The shifted DGZ coordinates are:

$X_{DGZ^{as}_I}$
$$= \beta_I X_{DGZ^a_I} + (1-\beta_I) x_{P_I} = \beta_I [\, \alpha_I(X_I + TX_I) + (1-\alpha_I)(X_2 + TX_2)\,] + (1-\beta_I) x_{P_I}$$

$Y_{DGZ^{as}_I}$
$$= \beta_I Y_{DGZ^a_I} + (1-\beta_I) y_{P_I} = \beta_I [\, \alpha_I(Y_I + TY_I) + (1-\alpha_I)(Y_2 + TY_2)\,] + (1-\beta_I) y_{P_I}$$

Thus $\mu_{x DGZ^{as}_I} = \beta_I [\, \alpha_I(\mu_{xI} + \mu_{txI}) + (1-\alpha_I)(\mu_{x2} + \mu_{tx2})\,] + (1-\beta_I) x_{P_I} \doteq -164$

and $\mu_{y DGZ^{as}_I} = \beta_I [\, \alpha_I(\mu_{yI} + \mu_{tyI}) + (1-\alpha_I)(\mu_{y2} + \mu_{ty2})\,] + (1-\beta_I) y_{P_I} \doteq 350$

(a) Unit #1:

$$X_{DGZ^{as}_I} - X_I = (\alpha_I\beta_I - 1) X_I + (1-\alpha_I)\beta_I X_2 + \alpha_I\beta_I TX_I + (1-\alpha_I)\beta_I TX_2 + (1-\beta) x_{P_I}$$

$$Y_{DGZ^{as}_I} - Y_I = (\alpha_I\beta_I - 1) Y_I + (1-\alpha_I)\beta_I Y_2 + \alpha_I\beta_I TY_I + (1-\alpha_I)\beta_I TY_2 + (1-\beta) y_{P_I}$$

Let $\mu_{x I DGZ^{as}_I} \equiv E[X_{DGZ^{as}_I} - X_I] = (\alpha_I\beta_I - 1)\mu_{xI} + (1-\alpha_I)\beta_I \mu_{x2} + \alpha_I\beta_I\mu_{txI} + (1-\alpha_I)\beta_I\mu_{tx2}$
$$+ (1-\beta_I) x_{P_I}$$

$$\mu_{y1DGZ_1^{as}} \equiv E[Y_{DGZ_1^{as}} - Y_1] = (\alpha_1\beta_1 - 1)\mu_{y1} + (1-\alpha_1)\beta_1\mu_{y2} + \alpha_1\beta_1\mu_{ty1} + (1-\alpha_1)\beta_1\mu_{ty2}$$
$$+ (1-\beta_1)y_{P_1}$$

$$\sigma_{x1DGZ_1^{as}}^2 \equiv Var[X_{DGZ_1^{as}} - X_1] = (\alpha_1\beta_1 - 1)^2\sigma_{x1}^2 + ((1-\alpha_1)\beta_1)^2\sigma_{x2}^2$$
$$+ 2(\alpha_1\beta_1 - 1)((1-\alpha_1)\beta_1)\rho_{x12}\sigma_{x1}\sigma_{x2} + (\alpha_1\beta_1)^2\sigma_{tx1}^2 + ((1-\alpha_1)\beta_1)^2\sigma_{tx1}^2$$

$$\sigma_{y1DGZ_1^{as}}^2 \equiv Var[Y_{DGZ_1^{as}} - Y_1] = (\alpha_1\beta_1 - 1)^2\sigma_{y1}^2 + ((1-\alpha_1)\beta_1)^2\sigma_{y2}^2$$
$$+ 2(\alpha_1\beta_1 - 1)((1-\alpha_1)\beta_1)\rho_{y12}\sigma_{y1}\sigma_{y2} + (\alpha_1\beta_1)^2\sigma_{ty1}^2 + ((1-\alpha_1)\beta_1)^2\sigma_{ty1}^2$$

$$\mu_{1DGZ_1^{as}} \equiv \mu_{x1DGZ_1^{as}}^2 + \mu_{y1DGZ_1^{as}}^2 + \sigma_{x1DGZ_1^{as}}^2 + \sigma_{y1DGZ_1^{as}}^2$$

$$\sigma_{1DGZ_1^{as}}^2 \equiv 2\sigma_{x1DGZ_1^{as}}^2(\sigma_{x1DGZ_1^{as}}^2 + 2\mu_{x1DGZ_1^{as}}^2) + 2\sigma_{y1DGZ_1^{as}}^2(\sigma_{y1DGZ_1^{as}}^2 + 2\mu_{y1DGZ_1^{as}}^2)$$

Evaluating the terms, recalling that $\alpha_1 = 0.598$ and $\beta_1 = 1.164$,

$$\mu_{x1DGZ_1^{as}} = ((0.598)(1.164) - 1)\,0 + (1-0.598)(1.164)\,0 + 0.598(1.164)\,0$$
$$+ (1-0.598)(1.164)\,0 + (1-1.164)\,1000 = -164$$

$$\mu_{y1DGZ_1^{as}} = ((0.598)(1.164) - 1)\,0 + (1-0.598)(1.164)\,1100 + 0.598(1.164)\,0$$
$$+ (1-0.598)(1.164)\,0 + (1-1.164)\,1000 = 351$$

$$\sigma_{x1DGZ_1^{as}}^2 = [(0.598)(1.164) - 1]^2(2000)^2 + [(1-0.598)(1.164)]^2(2000)^2$$
$$+ 2[(0.598)(1.164) - 1][(1-0.598)(1.164)](0.5)(2000)^2 + [0.598(1.164)]^2(75)^2$$
$$+ [(1-0.598)(1.164)]^2(75)^2 = 680,629$$

$$\sigma_{y1DGZ_1^{as}}^2 = [(0.598)(1.164) - 1]^2(200)^2 + [(1-0.598)(1.164)]^2(200)^2$$
$$+ 2[(0.598)(1.164) - 1][(1-0.598)(1.164)](0.9)(200)^2 + [0.598(1.164)]^2(100)^2$$
$$+ [(1-0.598)(1.164)]^2(100)^2 = 9,248$$

$$\mu_{1DGZ_1^{as}} = 844,527$$
$$\sigma_{1DGZ_1^{as}}^2 = 1.007 \times 10^{12}; \quad \sigma_{1DGZ_1^{as}} = 1,003,453$$

(b) *Unit #2:*

$$X_{DGZ_1^{as}} - X_2 = \alpha_1\beta_1 X_1 + (\beta_1 - \alpha_1\beta_1 - 1) X_2 + \alpha_1\beta_1 TX_1 + (1-\alpha_1)\beta_1 TX_2 + (1-\beta) x_{P_1}$$
$$Y_{DGZ_1^{as}} - Y_2 = \alpha_1\beta_1 Y_1 + (\beta_1 - \alpha_1\beta_1 - 1) Y_2 + \alpha_1\beta_1 TY_1 + (1-\alpha_1)\beta_1 TY_2 + (1-\beta) y_{P_1}$$

Let $\mu_{x2DGZ_1^{as}} \equiv E[X_{DGZ_1^{as}} - X_2] = \alpha_1\beta_1\mu_{x1} + (\beta_1 - \alpha_1\beta_1 - 1)\mu_{x2} + \alpha_1\beta_1\mu_{tx1} + (1-\alpha_1)\beta_1\mu_{tx2}$
$$+ (1-\beta_1) x_{P_1}$$

$$\mu_{y2DGZ_1^{as}} \equiv E[Y_{DGZ_1^{as}} - Y_2] = \alpha_1\beta_1\mu_{y1} + (\beta_1 - \alpha_1\beta_1 - 1)\mu_{y2} + \alpha_1\beta_1\mu_{ty1} + (1-\alpha_1)\beta_1\mu_{ty2}$$
$$+ (1-\beta_1) y_{P_1}$$

$$\sigma_{x2DGZ_1^{as}}^2 \equiv Var[X_{DGZ_1^{as}} - X_2] = (\alpha_1\beta_1)^2\sigma_{x1}^2 + (\beta_1 - \alpha_1\beta_1 - 1)^2\sigma_{x2}^2$$
$$+ 2(\alpha_1\beta_1)(\beta_1 - \alpha_1\beta_1 - 1)\rho_{x12}\sigma_{x1}\sigma_{x2} + (\alpha_1\beta_1)^2\sigma_{tx1}^2 + ((1-\alpha_1)\beta_1)^2\sigma_{tx1}^2$$

$$\sigma^2_{y2DGZ^{as}_l} \equiv \text{Var}[Y_{DGZ^{as}_l} - Y_2] = (\alpha_l\beta_l)^2\sigma^2_{yl} + (\beta_l - \alpha_l\beta_l - 1)^2\sigma^2_{y2}$$
$$+ 2(\alpha_l\beta_l)(\beta_l - \alpha_l\beta_l - 1)\rho_{y12}\sigma_{yl}\sigma_{y2} + (\alpha_l\beta_l)^2\sigma^2_{iyl} + ((1-\alpha_l)\beta_l)^2\sigma^2_{iyl}$$

$$\mu_{2DGZ^{as}_l} \equiv \mu^2_{x2DGZ^{as}_l} + \mu^2_{y2DGZ^{as}_l} + \sigma^2_{x2DGZ^{as}_l} + \sigma^2_{y2DGZ^{as}_l}$$

$$\sigma^2_{2DGZ^{as}_l} \equiv 2\sigma^2_{x2DGZ^{as}_l}(\sigma^2_{x2DGZ^{as}_l} + 2\mu^2_{x2DGZ^{as}_l}) + 2\sigma^2_{y2DGZ^{as}_l}(\sigma^2_{y2DGZ^{as}_l} + 2\mu^2_{y2DGZ^{as}_l})$$

Evaluating the terms, recalling that $\alpha_l = 0.598$ and $\beta_l = 1.164$,

$$\mu_{x2DGZ^{as}_l} = (0.598)(1.164)\,0 + (1.164 - (0.598)(1.164) - 1)\,0 + 0.598(1.164)\,0$$
$$+ (1-0.598)(1.164)\,0 + (1-1.164)\,1000 = -164$$

$$\mu_{y2DGZ^{as}_l} = (0.598)(1.164)\,0 + (1.164 - (0.598)(1.164) - 1)\,1100 + 0.598(1.164)\,0$$
$$+ (1-0.598)(1.164)\,0 + (1-1.164)\,1000 = -750$$

$$\sigma^2_{x2DGZ^{as}_l} = [(0.598)(1.164)]^2(2000)^2 + [(1.164 - (0.598)(1.164) - 1)]^2(2000)^2$$
$$+ 2[(0.598)(1.164)][1.164 - (0.598)(1.164) - 1](0.5)(2000)^2$$
$$+ [0.598(1.164)]^2(75)^2 + [(1-0.598)(1.164)]^2(75)^2 = 1,592,629$$

$$\sigma^2_{y2DGZ^{as}_l} = [(0.598)(1.164)]^2(200)^2 + [(1.164 - (0.598)(1.164) - 1)]^2(200)^2$$
$$+ 2[(0.598)(1.164)][1.164 - (0.598)(1.164) - 1](0.9)(200)^2$$
$$+ [0.598(1.164)]^2(100)^2 + [(1-0.598)(1.164)]^2(100)^2 = 11,712$$

$$\mu_{2DGZ^{as}_l} = 2,193,737$$

$$\sigma^2_{2DGZ^{as}_l} = 5.271 \times 10^{12}; \quad \sigma_{2DGZ^{as}_l} = 2,295,845$$

Recall that $p_{shift}(ij \mid w, a_{ij}, agg_{ij})$

$$= \left[\min_k \left\{ P[D^2_{DP_k} > r^2_{WP_k}U_{ij} \mid D^2_{ij} \le d^2_{ij}w] \right\} - \min_k \left\{ P[D^2_{DP_k} > r^2_{WP_k} \mid D^2_{ij} \le d^2_{ij}w] \right\} \right]$$
$$\cdot \frac{P[D^2_{iDGZ^{as}_l} \le d^2_i w \cap D^2_{jDGZ^{as}_l} \le d^2_j w]}{P[D^2_{ij} \le d^2_{ij}w]}$$

$$= \left[\overline{p}_{prec}(ij \mid w, a_{ij}, agg_{ij}) - p_{no\ shift}(ij \mid w, a_{ij}, agg_{ij}) \right]$$
$$\cdot \frac{P[D^2_{iDGZ^{as}_l} \le d^2_i w \cap D^2_{jDGZ^{as}_l} \le d^2_j w]}{p_{aggr}(ij \mid w, a_{ij})}$$

We know that (assuming independence, as usual, between $D^2_{iDGZ^{as}_l}$ and $D^2_{jDGZ^{as}_l}$)

$$P[D^2_{iDGZ^{as}_l} \le d^2_i w \cap D^2_{jDGZ^{as}_l} \le d^2_j w]$$

$$= 1 - P[D^2_{iDGZ^{as}_l} > d^2_i w] - P[D^2_{jDGZ^{as}_l} > d^2_j w] + P[D^2_{iDGZ^{as}_l} > d^2_i w]P[D^2_{jDGZ^{as}_l} > d^2_j w].$$

Let

$$h_1 \equiv P[D^2_{1\,DGZ^{as}_1} > d^2_{1\,W_1}] = P[\,Z > \frac{925^2 - \mu_{1\,DGZ^{as}_1}}{\sigma_{1\,DGZ^{as}_1}}\,] = P[\,Z > 0.011\,] = 0.496$$

$$k_1 \equiv P[D^2_{2\,DGZ^{as}_1} > d^2_{2\,W_1}] = P[\,Z > \frac{925^2 - \mu_{2\,DGZ^{as}_1}}{\sigma_{2\,DGZ^{as}_1}}\,] = P[\,Z \geq -0.583\,] = 0.720$$

Then $P[D^2_{1\,DGZ^{as}_1} \leq d^2_{1\,W_1} \cap D^2_{2\,DGZ^{as}_1} \leq d^2_{2\,W_1}] = 1 - h_1 - k_1 + h_1 k_1 = 0.141$

Thus

$$P_{shift}(12\,|\,w_1, a_{12}, agg_{12}) \,=\, \Big[\,\overline{P}_{prec}(12\,|\,w_1, a_{12}, agg_{12}) - P_{no\ shift}(12\,|\,w_1, a_{12}, agg_{12})\,\Big]$$

$$\cdot \frac{P[D^2_{1\,DGZ^{as}_1} \leq d^2_{1\,W_1} \cap D^2_{2\,DGZ^{as}_1} \leq d^2_{2\,W_1}]}{P_{aggr}(12\,|\,w_1, a_{12})}$$

$$= [\,0.669 - 0.353\,] \cdot \frac{0.141}{0.480} = 0.093.$$

Weapon #2:

Recall that $d_{1\,W_2} = d_{2\,W_2} = 1355$, and $E[\,D^2_{12}\,] = 9{,}249{,}250$. Then using the priority and target unit size criteria for establishing α_2. In this problem, $U = D^2_{12}$, $B = (\,d_{1\,W_2} + d_{2\,W_2})^2$, $A = 0$, $\mu = E[\,D^2_{12}\,]$ and $\sigma = \sqrt{\mathrm{Var}[\,D^2_{12}\,]}$. Thus

$$\hat{D}^2_{12} = E[D^2_{12}\,|\,D^2_{12} \leq (\,d_{1\,W_2} + d_{2\,W_2})^2] = 9{,}249{,}250 + \left[\frac{.286 - .393}{.433 - .207}\right](11{,}335{,}667)$$

$$= 3{,}882{,}363 \text{ and } \hat{D}_{12} = 1970$$

$$\alpha_2 = \min\Bigg\{ \frac{d_{2\,W_2}}{\hat{D}_{12}}\;;$$

$$\frac{\hat{D}_{12} - d_{1\,W_2}}{\hat{D}_{12}} + \frac{{}^r T_1}{{}^r T_1 + {}^r T_2}\left[\frac{d_{2\,W_2}}{\hat{D}_{12}} - \frac{\hat{D}_{12} - d_{1\,W_2}}{\hat{D}_{12}}\right] \cdot \left[1 + \frac{[2] - [1]}{\max\{\,[1],\,[2]\,\}}\right] \Bigg\}$$

Thus

$$\alpha_2 = \min\Bigg\{ 0.688;\ 0.312 + \tfrac{1}{2}[\,0.688 - 0.312\,][\,1 + \tfrac{1}{2}\,] \Bigg\} = \min\{\,0.688;\,0.594\,\} = 0.594$$

Thus the aggregate target DGZ^{a}_1 for weapon #2 has the following coordinates:

$$X_{DGZ^a_2} = \alpha_2 X_{1L} + (\,1 - \alpha_2)\,X_{2L};\ \mu_{x\,DGZ^a_2} = 0.594(0) + (1 - 0.594)(0) = 0$$

$$Y_{DGZ^a_2} = \alpha_2 Y_{1L} + (\,1 - \alpha_2)\,Y_{2L};\ \mu_{y\,DGZ^a_2} = 0.594(0) + (1 - 0.594)(1100) = 446.6 \doteq 447$$

NOTE: This DGZ is sufficiently close to the DGZ computed using weapon #1 that the same DGZ could be used in practice. However, for purposes of illustration, I will compute the various probabilities for the second weapon using the α_2 value of 0.594 with $(\mu_{x\,DGZ^a_2}, \mu_{y\,DGZ^a_2}) = (\,0, 447\,)$.

(1) *Computing the distributions between DGZ_2^a and preclusion areas 1 and 2:*

(a) *Preclusion area #1:*

We can adjust for the fact that the target units are aggregated, thus $D_{12}^2 \leq d_{12\,W_2}^2$.

$$\overline{P}_{prec}(ij|w_2, a_{ij}, agg_{ij}) = \min_k \left\{ P[\, D_{DP_k}^2 > r_{W_2P_kU_{ij}}^2 \,|\, D_{12}^2 \leq d_{12\,W_2}^2\,]\, \right\} \text{ and}$$

$$P_{no\ shift}(ij|w_2, a_{ij}, agg_{ij}) = \min_k \left\{ P[\, D_{DP_k}^2 > r_{W_2P_k}^2 \,|\, D_{12}^2 \leq d_{12\,W_2}^2\,]\, \right\}.$$

Recall that $E[D_{12}^2]$, $Var[D_{12}^2]$, $E[D_{1P_1}^2]$, $Var[D_{1P_1}^2]$, $E[D_{2P_1}^2]$, and $Var[D_{2P_1}^2]$ remain the same.

Then using the formula on page 46 with A=0, $B=d_{12\,W_2}^2=(2 \cdot 1355)^2$, $\mu=E[D_{12}^2]$ and $\sigma^2=Var[D_{12}^2]$,

$$E[D_{12}^2\,|\, D_{12}^2 \leq d_{12\,W_2}^2\,] = 9,249,250 + \left[\frac{.286-.393}{.433-.207}\right](11,335,667) = 3,882,363$$

$$Var[D_{12}^2\,|\, D_{12}^2 \leq d_{12\,W_2}^2] = \left\{ 1 + \left[\frac{(-.816)(.286)-(-.168)(.393)}{.433-.207}\right] - \left[\frac{.286-.393}{.433-.207}\right]^2 \right\} \cdot 11,335.667$$

$$= 0.036 \cdot 11,335,667^2 = 4.626 \times 10^{14} \approx 2,150,800^2$$

Thus for $\alpha_2 = 0.594$,

$$E[D_{DP_1}^2\,|\, D_{12}^2 \leq d_{12\,W_2}^2] = (\alpha_2{}^2-\alpha_2)E[D_{12}^2\,|\, D_{12}^2 \leq d_{12\,W_2}^2] + \alpha_2 E[D_{1P_1}^2] + (1-\alpha_2)E[D_{2P_1}^2]$$

$$= (-0.241)(3,882,363) + (0.594)(6,055,625) + (0.406)(5,065,625) = 4,717,399$$

$$Var[D_{DP_1}^2\,|\, D_{12}^2 \leq d_{12\,W_2}^2]$$

$$= (\alpha_2{}^2-\alpha_2)^2 Var[D_{12}^2\,|\, D_{12}^2 \leq d_{12\,W_2}^2] + \alpha_2{}^2 Var[D_{1P_k}^2] + (1-\alpha_2)^2 Var[D_{2P_k}^2]$$

$$= (-0.241)^2(2,150,800)^2 + (0.594)^2(6,951,084)^2 + (0.406)^2(6,936,826)^2$$

$$= 2.525 \times 10^{13}; \text{ with } \sqrt{Var[D_{DP_1}^2\,|\, D_{12}^2 \leq d_{12\,W_2}^2]} = 5,024,800$$

Thus

$$P[\, D_{DP_1}^2 > r_{W_2P_1U_{12}}^2 \,|\, D_{12}^2 \leq d_{12\,W_2}^2\,] = P[\, Z > \frac{1805^2 - 4,717,399}{5,024,800}\,] = P[\, Z > -0.290\,] = 0.614$$

$$P[\, D_{DP_1}^2 > r_{W_2P_1}^2 \,|\, D_{12}^2 \leq d_{12\,W_2}^2\,] = P[\, Z > \frac{3160^2 - 4,717,399}{5,024,800}\,] = P[\, Z > 1.048\,] = 0.147$$

(b) *Preclusion area #2:*

Recall that $E[D^2_{12} | D^2_{12} \leq d^2_{12\,W_2}]$, $\text{Var}[D^2_{12} | D^2_{12} \leq d^2_{12\,W_2}]$, $E[D^2_{1\,P_2}]$, $\text{Var}[D^2_{1\,P_2}]$, $E[D^2_{2\,P_2}]$, and $\text{Var}[D^2_{2\,P_2}]$ remain the same.

Thus for $\alpha_2 = 0.594$,

$$E[D^2_{D\,P_2} | D^2_{12} \leq d^2_{12\,W_2}] = (\alpha_2{}^2 - \alpha_2)E[D^2_{12} | D^2_{12} \leq d^2_{12\,W_2}] + \alpha_2 E[D^2_{1\,P_2}] + (1 - \alpha_2)E[D^2_{2\,P_2}]$$

$$= (-0.241)(3{,}882{,}363) + (0.594)(6{,}865{,}625) + (0.406)(4{,}555{,}625) = 4{,}992{,}116$$

$$\text{Var}[D^2_{D\,P_2} | D^2_{12} \leq d^2_{12\,W_2}]$$

$$= (\alpha_2{}^2 - \alpha_2)^2 \text{Var}[D^2_{12} | D^2_{12} \leq d^2_{12\,W_2}] + \alpha_2{}^2 \text{Var}[D^2_{1\,P_2}] + (1 - \alpha_2)^2 \text{Var}[D^2_{2\,P_2}]$$

$$= (-0.241)^2 (3{,}882{,}363)^2 + (0.594)^2 (6{,}865{,}625)^2 + (0.406)^2 (4{,}555{,}625)^2$$

$$= 1.915 \times 10^{13}; \text{ with } \sqrt{\text{Var}[D^2_{D\,P_2} | D^2_{12} \leq d^2_{12\,W_2}]} = 4{,}375{,}600$$

Thus

$$P[D^2_{D\,P_2} > r^2_{W_2 P_2 U_{12}} | D^2_{12} \leq d^2_{12\,W_2}] = P[Z > \frac{1305^2 - 4{,}992{,}116}{4{,}375{,}600}] = P[Z > -0.752] = 0.774$$

$$P[D^2_{D\,P_2} > r^2_{W_2 P_2} | D^2_{12} \leq d^2_{12\,W_2}] = P[Z > \frac{2660^2 - 4{,}992{,}116}{4{,}375{,}600}] = P[Z > 0.476] = 0.317$$

For weapon #2,

$$\overline{P}_{prec}(12 | w_2, a_{12}, agg_{12}) \quad = \min_k \left\{ P[D^2_{D\,P_k} > r^2_{W_2 P_k U_{12}} | D^2_{12} \leq d^2_{12\,W_2}] \right\}$$
$$= \min \left\{ 0.614, 0.774 \right\}$$
$$= 0.614$$

$$P_{no\ shift}(12 | w_2, a_{12}, agg_{12}) \quad = \min_k \left\{ P[D^2_{D\,P_k} > r^2_{W_2 P_k} | D^2_{12} \leq d^2_{12\,W_2}] \right\}.$$
$$= \min \left\{ 0.147, 0.317 \right\}$$
$$= 0.147$$

NOTE: The closest preclusion area for the aggregate target formed from units #1 and #2, weapon #2 is preclusion area #1, so all weapon #2 DGZ shifts will be computed from preclusion area #1.

(2) *Computing the probability of a DGZ_2^a shift:*

To evaluate the probability of a shift in the DGZ for the aggregate target formed from target units #1 and #2, engaged with weapon type #2, we need to calculate the distribution of the shifted DGZ.

We begin by evaluating β_2. To do this, we need to evaluate

$$E[\ D_{DP_1}^2 \,|\, D_{12}^2 \le (d_1 w_2 + d_2 w_2)^2 \cap \alpha_2 r_{W_2 P_1 U_1}^2 + (1-\alpha_2) r_{W_2 P_1 U_2}^2$$

$$\le \alpha_2 D_{1 P_1}^2 + (1-\alpha_2) D_{2 P_1}^2 < r_{W_2 P_1}^2 + (\alpha_2 - \alpha_1^2)(d_1 w_2 + d_2 w_2)^2\ ]$$

$$\doteq (\alpha_1^2 - \alpha_2) E[D_{12}^2 \,|\, D_{12}^2 \le (d_1 w_2 + d_2 w_2)^2] \ + \ E[\!\![\ \alpha_2 D_{1 P_1}^2 + (1-\alpha_2) D_{2 P_1}^2 \,|\, \alpha_2 r_{W_2 P_1 U_1}^2$$

$$+ (1-\alpha_2) r_{W_2 P_1 U_2}^2 \le \alpha_2 D_{1 P_1}^2 + (1-\alpha_2) D_{2 P_1}^2 < r_{W_2 P_1}^2 + (\alpha_2 - \alpha_1^2)(d_1 w_2 + d_2 w_2)^2\ ]\!\!]$$

$E[\ D_{12}^2 \,|\, D_{12}^2 \le (d_1 w_2 + d_2 w_2)^2 \,]$ has been evaluated previously as 3,882,363.

Recall that

$$\mu_{1 P_1} = 6{,}055{,}625 \qquad\qquad \mu_{2 P_1} = 5{,}065{,}625$$

$$\sigma_{1 P_1} = 6{,}951{,}084 \qquad\qquad \sigma_{2 P_1} = 6{,}936{,}826$$

Then $\alpha_1 \mu_{1 P_1} + (1-\alpha_1) \mu_{2 P_1} = 5{,}653{,}685$

$$\alpha_1^2 \sigma_{1 P_1}^2 + (1-\alpha_1)^2 \sigma_{2 P_1}^2 = 2.491 \times 10^{13};\ \text{square root} = 4{,}998{,}000$$

Let $U \equiv \alpha_1 D_{1 P_1}^2 + (1-\alpha_1) D_{2 P_1}^2$; $\mu_U = 5{,}653{,}685$; $\sigma_U = 4{,}998{,}000$;

$$A \equiv \alpha_1 r_{W_2 P_1 U_1}^2 + (1-\alpha_1) r_{W_2 P_1 U_2}^2;\ B \equiv r_{W_2 P_1}^2 + (\alpha_1 - \alpha_1^2)(d_1 w_2 + d_2 w_2)^2;\ \text{then}$$

Then $E[\ U \,|\, A \le U < B\ ] = 5{,}653{,}685 + \left[\dfrac{.356 - .189}{.889 - .316} \right](4{,}998{,}000) = 7{,}110{,}345.$

Thus $E[\ D_{D P_1}^2 \,|\, D_{12}^2 \le (d_1 w_2 + d_2 w_2)^2 \cap A \le U < B\ ]$

$$\doteq (0.594^2 - 0.594)(3{,}882{,}363) + 7{,}110{,}345 = 6{,}174{,}059$$

Recalling that $r_{W_2 P_1} = 3160$, and defining

$$\dot{D}_{D P_1} \equiv \sqrt{E[\ D_{D P_1}^2 \,|\, D_{12}^2 \le (d_1 w_2 + d_2 w_2)^2 \cap A \le U < B\ ]},\ \text{then}\ \beta_2 = \frac{r_{W_2 P_1}}{\dot{D}_{D P_1}} = 1.272.$$

The shifted DGZ coordinates are:

$$X_{DGZ_2^a}^s$$
$$= \beta_2 X_{DGZ_2^a} + (1-\beta_2) x_{P_1} = \beta_2 [\ \alpha_2 (X_1 + TX_1) + (1-\alpha_2)(X_2 + TX_2)\ ] + (1-\beta_2) x_{P_1}$$
$$Y_{DGZ_2^a}^s$$
$$= \beta_2 Y_{DGZ_2^a} + (1-\beta_2) y_{P_1} = \beta_2 [\ \alpha_2 (Y_1 + TY_1) + (1-\alpha_2)(Y_2 + TY_2)\ ] + (1-\beta_2) y_{P_1}$$

Thus $\mu_{x\,DGZ_2^{as}} = \beta_2\,[\,\alpha_2(\,\mu_{x1}+\mu_{tx1})\,+(1-\alpha_2)(\,\mu_{x2}+\mu_{tx2})\,]\,+\,(\,1-\beta_2\,)\,\mathrm{x}_{P_1} \doteq -272$

and $\mu_{y\,DGZ_2^{as}} = \beta_2\,[\,\alpha_2(\,\mu_{y1}+\mu_{ty1})\,+(1-\alpha_2)(\,\mu_{y2}+\mu_{ty2})\,]\,+\,(\,1-\beta_2\,)\,\mathrm{y}_{P_1} \doteq 296$

(a) *Unit #1:*

$$X_{DGZ_2^{as}} - X_1 = (\alpha_2\beta_2-1)\,X_1\,+(1-\alpha_2)\beta_2\,X_2\,+\,\alpha_2\beta_2\,TX_1\,+(1-\alpha_2)\beta_2\,TX_2\,+\,(\,1-\beta\,)\,\mathrm{x}_{P_1}$$

$$Y_{DGZ_2^{as}} - Y_1 = (\alpha_2\beta_2-1)\,Y_1\,+(1-\alpha_2)\beta_2\,Y_2\,+\,\alpha_2\beta_2\,TY_1\,+(1-\alpha_2)\beta_2\,TY_2\,+\,(\,1-\beta\,)\,\mathrm{y}_{P_1}$$

Let $\mu_{x1\,DGZ_2^{as}} \equiv \mathrm{E}[X_{DGZ_2^{as}}-X_1] = (\alpha_2\beta_2-1)\,\mu_{x1}\,+(1-\alpha_2)\beta_2\,\mu_{x2}\,+\,\alpha_2\beta_2\mu_{tx1}$

$\qquad\qquad + (1-\alpha_2)\beta_2\mu_{tx2}\,+\,(\,1-\beta_2\,)\,\mathrm{x}_{P_1}$

$\mu_{y1\,DGZ_2^{as}} \equiv \mathrm{E}[Y_{DGZ_2^{as}}-Y_1] = (\alpha_2\beta_2-1)\,\mu_{y1}\,+(1-\alpha_2)\beta_2\,\mu_{y2}\,+\,\alpha_2\beta_2\mu_{ty1}$

$\qquad\qquad + (1-\alpha_2)\beta_2\mu_{ty2}\,+\,(\,1-\beta_2\,)\,\mathrm{y}_{P_1}$

$\sigma_{x1\,DGZ_2^{as}}^2 \equiv \mathrm{Var}[X_{DGZ_2^{as}}-X_1] = (\alpha_2\beta_2-1)^2\sigma_{x1}^2\,+\,((1-\alpha_2)\beta_2)^2\sigma_{x2}^2$

$\qquad\qquad + 2(\alpha_2\beta_2-1)((1-\alpha_2)\beta_2)\rho_{x12}\sigma_{x1}\sigma_{x2}\,+\,(\alpha_2\beta_2)^2\sigma_{tx1}^2\,+\,((1-\alpha_2)\beta_2)^2\sigma_{tx2}^2$

$\sigma_{y1\,DGZ_2^{as}}^2 \equiv \mathrm{Var}[Y_{DGZ_2^{as}}-Y_1] = (\alpha_2\beta_2-1)^2\sigma_{y1}^2\,+\,((1-\alpha_2)\beta_2)^2\sigma_{y2}^2$

$\qquad\qquad + 2(\alpha_2\beta_2-1)((1-\alpha_2)\beta_2)\rho_{y12}\sigma_{y1}\sigma_{y2}\,+\,(\alpha_2\beta_2)^2\sigma_{ty1}^2\,+\,((1-\alpha_2)\beta_2)^2\sigma_{ty2}^2$

$\mu_{1\,DGZ_2^{as}} \equiv \mu_{x1\,DGZ_2^{as}}^2\,+\,\mu_{y1\,DGZ_2^{as}}^2\,+\,\sigma_{x1\,DGZ_2^{as}}^2\,+\,\sigma_{y1\,DGZ_2^{as}}^2$

$\sigma_{1\,DGZ_2^{as}}^2 \equiv 2\sigma_{x1\,DGZ_2^{as}}^2(\sigma_{x1\,DGZ_2^{as}}^2+2\mu_{x1\,DGZ_2^{as}}^2)\,+\,2\sigma_{y1\,DGZ_2^{as}}^2(\sigma_{y1\,DGZ_2^{as}}^2+2\mu_{y1\,DGZ_2^{as}}^2)$

Evaluating the terms, recalling that $\alpha_2 = 0.594$ and $\beta_2 = 1.272$,

$\mu_{x1\,DGZ_2^{as}} = (\,(0.594)(1.272)-1\,)\,0\,+\,(1-0.594)(1.272)\,0\,+\,0.594(1.272)\,0$

$\qquad\qquad + (1-0.594)(1.272)\,0\,+\,(1-1.272)\,1000\,=\,-272$

$\mu_{y1\,DGZ_2^{as}} = (\,(0.594)(1.272)-1\,)\,0\,+\,(1-0.594)(1.272)\,1100\,+\,0.594(1.272)\,0$

$\qquad\qquad + (1-0.594)(1.272)\,0\,+\,(1-1.272)\,1000\,=\,295.6$

$\sigma_{x1\,DGZ_2^{as}}^2 = [(0.594)(1.272)-1]^2\,(2000)^2\,+\,[(1-0.594)(1.272)]^2\,(2000)^2$

$\qquad\qquad + 2[(0.594)(1.272)-1][(1-0.594)(1.272)](0.5)\,(2000)^2$

$\qquad\qquad + [0.594(1.272)]^2\,(75)^2\,+\,[(1-0.594)(1.272)]^2\,(75)^2\,=\,804{,}265$

$\sigma_{y1\,DGZ_2^{as}}^2 = [(0.594)(1.272)-1]^2\,(200)^2\,+\,[(1-0.594)(1.272)]^2\,(200)^2$

$\qquad\qquad + 2[(0.594)(1.272)-1][(1-0.594)(1.272)](0.9)\,(200)^2$

$\qquad\qquad + [0.594(1.272)]^2\,(100)^2\,+\,[(1-0.594)(1.272)]^2\,(100)^2\,=\,12{,}345$

$\mu_{1\,DGZ_2^{as}} = 977{,}972$

$\sigma_{1\,DGZ_2^{as}}^2 = 1.536 \times 10^{12};\quad \sigma_{1\,DGZ_2^{as}} = 1{,}239{,}480.$

(b) *Unit #2:*

$$X_{DGZ_2^{as}} - X_2 = \alpha_2\beta_2 X_1 + (\beta_2 - \alpha_2\beta_2 - 1) X_2 + \alpha_2\beta_2 TX_1 + (1-\alpha_2)\beta_2 TX_2 + (1-\beta) x_{P_1}$$

$$Y_{DGZ_2^{as}} - Y_2 = \alpha_2\beta_2 Y_1 + (\beta_2 - \alpha_2\beta_2 - 1) Y_2 + \alpha_2\beta_2 TY_1 + (1-\alpha_2)\beta_2 TY_2 + (1-\beta) y_{P_1}$$

Let $\mu_{x2DGZ_2^{as}} \equiv E[X_{DGZ_2^{as}} - X_2] = \alpha_2\beta_2\mu_{x1} + (\beta_2 - \alpha_2\beta_2 - 1)\mu_{x2} + \alpha_2\beta_2\mu_{tx1}$
$$+ (1-\alpha_2)\beta_2\mu_{tx2} + (1-\beta_2) x_{P_1}$$

$\mu_{y2DGZ_2^{as}} \equiv E[Y_{DGZ_2^{as}} - Y_2] = \alpha_2\beta_2\mu_{y1} + (\beta_2 - \alpha_2\beta_2 - 1)\mu_{y2} + \alpha_2\beta_2\mu_{ty1}$
$$+ (1-\alpha_2)\beta_2\mu_{ty2} + (1-\beta_2) y_{P_1}$$

$\sigma^2_{x2DGZ_2^{as}} \equiv Var[X_{DGZ_2^{as}} - X_2] = (\alpha_2\beta_2)^2\sigma^2_{x1} + (\beta_2 - \alpha_2\beta_2 - 1)^2\sigma^2_{x2}$
$$+ 2(\alpha_2\beta_2)(\beta_2 - \alpha_2\beta_2 - 1)\rho_{x12}\sigma_{x1}\sigma_{x2} + (\alpha_2\beta_2)^2\sigma^2_{tx1} + ((1-\alpha_2)\beta_2)^2\sigma^2_{tx2}$$

$\sigma^2_{y2DGZ_2^{as}} \equiv Var[Y_{DGZ_2^{as}} - Y_2] = (\alpha_2\beta_2)^2\sigma^2_{y1} + (\beta_2 - \alpha_2\beta_2 - 1)^2\sigma^2_{y2}$
$$+ 2(\alpha_2\beta_2)(\beta_2 - \alpha_2\beta_2 - 1)\rho_{y12}\sigma_{y1}\sigma_{y2} + (\alpha_2\beta_2)^2\sigma^2_{ty1} + ((1-\alpha_2)\beta_2)^2\sigma^2_{ty2}$$

$\mu_{2DGZ_2^{as}} \equiv \mu^2_{x2DGZ_2^{as}} + \mu^2_{y2DGZ_2^{as}} + \sigma^2_{x2DGZ_2^{as}} + \sigma^2_{y2DGZ_2^{as}}$

$\sigma^2_{2DGZ_2^{as}} \equiv 2\sigma^2_{x2DGZ_2^{as}}(\sigma^2_{x2DGZ_2^{as}} + 2\mu^2_{x2DGZ_2^{as}}) + 2\sigma^2_{y2DGZ_2^{as}}(\sigma^2_{y2DGZ_2^{as}} + 2\mu^2_{y2DGZ_2^{as}})$

Evaluating the terms, recalling that $\alpha_2 = 0.594$ and $\beta_2 = 1.272$,

$\mu_{x2DGZ_2^{as}} = (0.594)(1.272)0 + (1.272 - (0.594)(1.272) - 1)0 + 0.594(1.272)0$
$$+ (1-0.594)(1.272)0 + (1-1.272)1000 = -272$$

$\mu_{y2DGZ_2^{as}} = (0.594)(1.272)0 + (1.272 - (0.594)(1.272) - 1)1100 + 0.594(1.272)0$
$$+ (1-0.594)(1.272)0 + (1-1.272)1000 = -804.4$$

$\sigma^2_{x2DGZ_2^{as}} = [(0.594)(1.272)]^2(2000)^2 + [(1.272 - (0.594)(1.272) - 1)]^2(2000)^2$
$$+ 2[(0.594)(1.272)][1.272 - (0.594)(1.272) - 1](0.5)(2000)^2$$
$$+ [0.594(1.272)]^2(75)^2 + [(1-0.594)(1.272)]^2(75)^2 = 1,764,265$$

$\sigma^2_{y2DGZ_2^{as}} = [(0.594)(1.272)]^2(200)^2 + [(1.272 - (0.594)(1.272) - 1)]^2(200)^2$
$$+ 2[(0.594)(1.272)][1.272 - (0.594)(1.272) - 1](0.9)(200)^2$$
$$+ [0.594(1.272)]^2(100)^2 + [(1-0.594)(1.272)]^2(100)^2 = 14,264$$

$\mu_{2DGZ_2^{as}} = 2,499,572$

$\sigma^2_{2DGZ_2^{as}} = 6.785 \times 10^{12}; \quad \sigma_{2DGZ_2^{as}} = 2,604,745$

$h_2 \equiv P[D^2_{1DGZ_2^{as}} > d^2_1 W_2] = P\left[Z > \dfrac{1355^2 - \mu_{1DGZ_1^{as}}}{\sigma_{1DGZ_1^{as}}}\right] = P[Z > 0.692] = 0.244$

$k_2 \equiv P[D^2_{2DGZ_1^{as}} > d^2_2 W_2] = P\left[Z > \dfrac{1355^2 - \mu_{2DGZ_1^{as}}}{\sigma_{2DGZ_1^{as}}}\right] = P[Z \geq -0.255] = 0.601$

Then $P[D^2_{1DGZ_1^{as}} \leq d^2_1 W_2 \cap D^2_{2DGZ_1^{as}} \leq d^2_2 W_2] = 1 - h_1 - k_1 + h_1 k_1 = 0.302.$

Thus

$$P_{shift}(12 \mid w_2, a_{12}, agg_{12}) = \left[\overline{P}_{prec}(12 \mid w_2, a_{12}, agg_{12}) - P_{no\ shift}(12 \mid w_2, a_{12}, agg_{12}) \right]$$

$$\cdot \frac{P[D^2_{1\,DGZ_2^{a,s}} \le d_1^2\, w_2 \cap D^2_{2\,DGZ_2^{a,s}} \le d_2^2\, w_2]}{P_{aggr}(12 \mid w_2, a_{12})}$$

$$= [\, 0.614 - 0.147\,] \cdot \frac{0.302}{0.616} = 0.229.$$

Computing the AGZ and conditional defeat distributions of the aggregate target

Let AGZ_1 denote the AGZ from firing the weapon of type 1 and AGZ_2 denote the AGZ from firing the weapon of type 2.

(1) *No DGZ shift, Weapon type 1:*

$$X_{AGZ_1} = X_{DGZ_1^a} + CX_1 = \alpha_1(X_1 + TX_1) + (1-\alpha_1)(X_2 + TX_2) + CX_1$$
$$= \alpha_1 X_1 + (1-\alpha_1) X_2 + \alpha_1 TX_1 + (1-\alpha_1)\, TX_2 + CX_1,$$
$$Y_{AGZ_1} = \alpha_1 Y_1 + (1-\alpha_1) Y_2 + \alpha_1 TY_1 + (1-\alpha_1) TY_2 + CY_1. \text{ Thus}$$

(a) *Unit #1:*

$$X_{AGZ_1} - X_1 = (\alpha_1-1) X_1 + (1-\alpha_1) X_2 + \alpha_1 TX_1 + (1-\alpha_1) TX_2 + CX_1 \text{ and}$$
$$Y_{AGZ_1} - Y_1 = (\alpha_1-1) Y_1 + (1-\alpha_1) Y_2 + \alpha_1 TY_1 + (1-\alpha_1) TY_2 + CY_1.$$

Let
$$\mu_{x1\,AGZ_1} \equiv E[X_{AGZ_1}-X_1] = (\alpha_1-1)\mu_{x1} + (1-\alpha_1)\mu_{x2} + \alpha_1\mu_{tx1} + (1-\alpha_1)\mu_{tx2} + \mu_{cx1}$$

$$\mu_{y1\,AGZ_1} \equiv E[Y_{AGZ_1}-Y_1] = (\alpha_1-1)\mu_{y1} + (1-\alpha_1)\mu_{y2} + \alpha_1\mu_{ty1} + (1-\alpha_1)\mu_{ty2} + \mu_{cy1}$$

$$\sigma^2_{x1\,AGZ_1} \equiv \text{Var}[X_{AGZ_1}-X_1] = (\alpha_1-1)^2\sigma^2_{x1} + (1-\alpha_1)^2\sigma^2_{x2} + 2(\alpha_1-1)(1-\alpha_1)\rho_{x12}\sigma_{x1}\sigma_{x2}$$
$$+ \alpha_1^2\sigma^2_{tx1} + (1-\alpha_1)^2\sigma^2_{tx1} + \sigma^2_{cx1}$$

$$\sigma^2_{y1\,AGZ_1} \equiv \text{Var}[Y_{AGZ_1}-Y_1] = (\alpha_1-1)^2\sigma^2_{y1} + (1-\alpha_1)^2\sigma^2_{y2} + 2(\alpha_1-1)(1-\alpha_1)\rho_{y12}\sigma_{y1}\sigma_{y2}$$
$$+ \alpha_1^2\sigma^2_{ty1} + (1-\alpha_1)^2\sigma^2_{ty1} + \sigma^2_{cy1}$$

$$\mu_{1\,AGZ_1} \equiv \mu^2_{x1\,AGZ_1} + \mu^2_{y1\,AGZ_1} + \sigma^2_{x1\,AGZ_1} + \sigma^2_{y1\,AGZ_1}$$

$$\sigma^2_{1\,AGZ_1} \equiv 2\sigma^2_{x1\,AGZ_1}(\sigma^2_{x1\,AGZ_1}+2\mu^2_{x1\,AGZ_1}) + 2\sigma^2_{y1\,AGZ_1}(\sigma^2_{y1\,AGZ_1}+2\mu^2_{y1\,AGZ_1})$$

Then the squared distance between the AGZ for weapon of type #1 and the unit #1, $D^2_{1\,AGZ_1}$, is:

$$D^2_{1\,AGZ_1} = D^2_{x\,1\,AGZ_1} + D^2_{y\,1\,AGZ_1} \dot{\sim} N[\mu_{1\,AGZ_1}, \sigma^2_{1\,AGZ_1}]$$

Evaluating the terms,

$$\mu_{x\,1\,AGZ_1} = (-0.402)0 + (0.402)0 + (0.598)0 + (0.402)0 + 0 = 0$$

$$\mu_{y\,1\,AGZ_1} = (-0.402)0 + (0.402)1100 + (0.598)0 + (0.402)0 + 0 = 442.2$$

$$\sigma^2_{x\,1\,AGZ_1} = (-0.402)^2 2000^2 + (0.402)^2 2000^2 + 2(-0.402)(0.402)(0.5)2000^2$$
$$+ (0.598)^2 75^2 + (0.402)^2 75^2 + 225^2 = 699{,}962$$

$$\sigma^2_{y\,1\,AGZ_1} = (-0.402)^2 200^2 + (0.402)^2 200^2 + 2(-0.402)(0.402)(0.9)200^2$$
$$+ (0.598)^2 100^2 + (0.402)^2 100^2 + 225^2 = 57{,}110$$

$$\mu_{1\,AGZ_1} = 952{,}613$$

$$\sigma^2_{1\,AGZ_1} = 1.031 \times 10^{12}; \quad \sigma_{1\,AGZ_1} = 1{,}015{,}423$$

The conditional probability of defeat of unit #1 as part of an aggregate target (with unit #2) given weapon type 1 and no DGZ shift is:

$$P_{defeat}(1\,|\,w_1, a_{12}, \overline{s}, agg_{12}) = P[\,D^2_{1\,AGZ_1} \le d^2_{1\,W_1}\,] = P[\,Z \le \frac{925^2 - \mu_{1\,AGZ_1}}{\sigma_{1\,AGZ_1}}\,] = P[\,Z \le -0.096\,]$$
$$= 0.462$$

(b) *Unit #2:*

$$X_{AGZ_1} - X_2 = \alpha_1 X_1 - \alpha_1 X_2 + \alpha_1 TX_1 + (1-\alpha_1) TX_2 + CX_1 \text{ and}$$
$$Y_{AGZ_1} - Y_2 = \alpha_1 Y_1 - \alpha_1 Y_2 + \alpha_1 TY_1 + (1-\alpha_1) TY_2 + CY_1.$$

Let $\mu_{x\,2\,AGZ_1} \equiv E[X_{AGZ_1} - X_2] = \alpha_1 \mu_{x1} - \alpha_1 \mu_{x2} + \alpha_1 \mu_{tx1} + (1-\alpha_1)\mu_{tx2} + \mu_{cx1}$

$\mu_{y\,2\,AGZ_1} \equiv E[Y_{AGZ_1} - Y_2] = \alpha_1 \mu_{y1} - \alpha_1 \mu_{y2} + \alpha_1 \mu_{ty1} + (1-\alpha_1)\mu_{ty2} + \mu_{cy1}$

$\sigma^2_{x\,2\,AGZ_1} \equiv Var[X_{AGZ_1} - X_2] = \alpha_1^2 \sigma^2_{x1} + (-\alpha_1)^2 \sigma^2_{x2} + 2(\alpha_1)(-\alpha_1)\rho_{x12}\sigma_{x1}\sigma_{x2}$
$\qquad\qquad + \alpha_1^2 \sigma^2_{tx1} + (1-\alpha_1)^2 \sigma^2_{tx1} + \sigma^2_{cx1}$

$\sigma^2_{y\,2\,AGZ_1} \equiv Var[Y_{AGZ_1} - Y_2] = \alpha_1^2 \sigma^2_{y1} + (-\alpha_1)^2 \sigma^2_{y2} + 2(\alpha_1)(-\alpha_1)\rho_{y12}\sigma_{y1}\sigma_{y2}$
$\qquad\qquad + \alpha_1^2 \sigma^2_{ty1} + (1-\alpha_1)^2 \sigma^2_{ty1} + \sigma^2_{cy1}$

$\mu_{2\,AGZ_1} \equiv \mu^2_{x\,2\,AGZ_1} + \mu^2_{y\,2\,AGZ_1} + \sigma^2_{x\,2\,AGZ_1} + \sigma^2_{y\,2\,AGZ_1}$

$\sigma^2_{2\,AGZ_1} \equiv 2\sigma^2_{x\,2\,AGZ_1}(\sigma^2_{x\,2\,AGZ_1} + 2\mu^2_{x\,2\,AGZ_1}) + 2\sigma^2_{y\,2\,AGZ_1}(\sigma^2_{y\,2\,AGZ_1} + 2\mu^2_{y\,2\,AGZ_1})$

Then the squared distance between the AGZ for weapon of type w and the unit 2, $D^2_{2\,AGZ_w}$, is:

$$D^2_{2\,AGZ_1} = D^2_{x\,2\,AGZ_1} + D^2_{y\,2\,AGZ_1} \dot{\sim} N[\mu_{2\,AGZ_1}, \sigma^2_{2\,AGZ_1}]$$

Evaluating the terms,

$$\mu_{x\,2\,AGZ_1} = (0.598)\,0 + (-0.598)\,0 + (0.598)\,0 + (0.402)\,0 + 0 = 0$$

$$\mu_{y\,2\,AGZ_1} = (0.598)\,0 + (-0.598)\,1100 + (0.598)\,0 + (0.402)\,0 + 0 = 657.8$$

$$\sigma^2_{x\,2\,AGZ_1} = (0.598)^2\,2000^2 + (-0.598)^2\,2000^2 + 2(0.598)(-0.598)(0.5)2000^2$$
$$+ (0.598)^2\,75^2 + (0.402)^2\,75^2 + 225^2 = 1{,}483{,}962$$

$$\sigma^2_{y\,2\,AGZ_1} = (-0.598)^2\,200^2 + (0.598)^2\,200^2 + 2(0.598)(-0.598)(0.9)200^2$$
$$+ (0.598)^2\,100^2 + (0.402)^2\,100^2 + 225^2 = 58{,}678$$

$$\mu_{2\,AGZ_1} = 1{,}975{,}340$$

$$\sigma^2_{2\,AGZ_1} = 4.513 \times 10^{12}; \quad \sigma_{2\,AGZ_1} = 2{,}124{,}320$$

The conditional probability of defeat of unit #2 as part of an aggregate target (with unit #1) given weapon type 1 and no DGZ shift is:

$$P_{defeat}(2\,|\,w_1, a_{12}, \overline{s}, agg_{12}) = P[\,D^2_{2\,AGZ_1} \leq d^2_{2\,W_1}\,] = P[\,Z \leq \frac{925^2 - \mu_{2\,AGZ_1}}{\sigma_{2\,AGZ_1}}\,] = P[\,Z \leq -0.527\,]$$
$$= 0.299$$

(2) *No DGZ shift, Weapon type 2:*

The squared distance between the AGZ for weapon of type #2 and unit i, $D^2_{i\,AGZ_2}$, is evaluated as before.

(a) *Unit #1:*

$$\mu_{x\,1\,AGZ_2} = (-0.406)\,0 + (0.406)\,0 + (0.594)\,0 + (0.406)\,0 + 0 = 0$$

$$\mu_{y\,1\,AGZ_2} = (-0.406)\,0 + (0.406)\,1100 + (0.594)\,0 + (0.406)\,0 + 0 = 446.6$$

$$\sigma^2_{x\,1\,AGZ_2} = (-0.406)^2\,2000^2 + (0.406)^2\,2000^2 + 2(-0.406)(0.406)(0.5)2000^2$$
$$+ (0.594)^2\,75^2 + (0.406)^2\,75^2 + 225^2 = 712{,}881$$

$$\sigma^2_{y\,1\,AGZ_2} = (-0.406)^2\,200^2 + (0.406)^2\,200^2 + 2(-0.406)(0.406)(0.9)200^2$$
$$+ (0.594)^2\,100^2 + (0.406)^2\,100^2 + 225^2 = 57{,}120$$

$$\mu_{1\,AGZ_2} = 969{,}453$$

$$\sigma^2_{1\,AGZ_2} = 1.068 \times 10^{12}; \quad \sigma_{1\,AGZ_2} = 1{,}033{,}680$$

The conditional probability of defeat of unit #1 as part of an aggregate target (with unit #2) given weapon type 2 and no DGZ shift is:

$$P_{defeat}(1 \mid w_2, a_{12}, \overline{s}, agg_{12}) = P[D^2_{1 AGZ_2} \leq d^2_1 w_2] = P[Z \leq \frac{1355^2 - \mu_{1 AGZ_2}}{\sigma_{1 AGZ_2}}] = P[Z \leq 0.858]$$
$$= 0.799$$

(b) *Unit #2:*

$$\mu_{x 2 AGZ_2} = (0.594)0 + (-0.594)0 + (0.594)0 + (0.406)0 + 0 = 0$$
$$\mu_{y 2 AGZ_2} = (0.594)0 + (-0.594)1100 + (0.594)0 + (0.406)0 + 0 = -653.4$$
$$\sigma^2_{x 2 AGZ_2} = 712,881$$
$$\sigma^2_{y 2 AGZ_2} = 57,120$$
$$\mu_{2 AGZ_2} = 1,196,933$$
$$\sigma^2_{2 AGZ_2} = 1.120 \times 10^{12}; \quad \sigma_{2 AGZ_2} = 1,058,522$$

The conditional probability of defeat of unit #2 as part of an aggregate target (with unit #1) given weapon type 2 and no DGZ shift is:

$$P_{defeat}(2 \mid w_2, a_{12}, \overline{s}, agg_{12}) = P[D^2_{2 AGZ_2} \leq d^2_2 w_1] = P[Z \leq \frac{1355^2 - \mu_{2 AGZ_2}}{\sigma_{2 AGZ_2}}] = P[Z \leq 0.604]$$
$$= 0.727$$

(3) *DGZ shift, Weapon type 1:*

We have previously evaluated the distribution of the shifted DGZ for the aggregate target, with random coordinates ($X_{DGZ^{as}_1}$, $Y_{DGZ^{as}_1}$). The AGZ coordinates ($X_{AGZ^s_1}$, $Y_{AGZ^s_1}$) may simply be determined by adding the random variables for the CEP shift in the X and Y directions (CX_1 and CY_1 respectively) to the X and Y coordinates of the DGZ. Thus

$$X_{AGZ^s_1} = X_{DGZ^{as}_1} + CX_1 \quad \text{and} \quad Y_{AGZ^s_1} = Y_{DGZ^{as}_1} + CY_1.$$

(a) *Unit #1:*

$$X_{AGZ^s_1} - X_1$$
$$= (\alpha_1 \beta_1 - 1) X_1 + (1 - \alpha_1)\beta_1 X_2 + \alpha_1 \beta_1 TX_1 + (1 - \alpha_1)\beta_1 TX_2 + (1 - \beta) x_{P_1} + CX_1$$
$$= X_{DGZ^{as}_1} - X_1 + CX_1$$

$$Y_{AGZ_I^s} - Y_I$$
$$= (\alpha_I \beta_I - 1) Y_I + (1 - \alpha_I)\beta_I Y_2 + \alpha_I \beta_I TY_I + (1 - \alpha_I)\beta_I TY_2 + (1 - \beta) y_{P_I} + CY_I$$
$$= Y_{DGZ_I^{as}} - Y_I + CY_I$$

Let

$$\mu_{x I AGZ_I^s} \equiv (\alpha_I \beta_I - 1) \mu_{xI} + (1 - \alpha_I)\beta_I \mu_{x2} + \alpha_I \beta_I \mu_{txI} + (1 - \alpha_I)\beta_I \mu_{tx2} + (1 - \beta_I) x_{P_I} + \mu_{cxI}$$
$$= \mu_{x I DGZ_I^{as}} + \mu_{cxI}$$

$$\mu_{y I AGZ_I^s} \equiv (\alpha_I \beta_I - 1) \mu_{yI} + (1 - \alpha_I)\beta_I \mu_{y2} + \alpha_I \beta_I \mu_{tyI} + (1 - \alpha_I)\beta_I \mu_{ty2} + (1 - \beta_I) y_{P_I} + \mu_{cyI}$$
$$= \mu_{y I DGZ_I^{as}} + \mu_{cyI}$$

$$\sigma_{x I AGZ_I^s}^2 \equiv \mathrm{Var}[X_{AGZ_I^s} - X_I] = (\alpha_I \beta_I - 1)^2 \sigma_{xI}^2 + ((1 - \alpha_I)\beta_I)^2 \sigma_{x2}^2$$
$$+ 2(\alpha_I \beta_I - 1)((1 - \alpha_I)\beta_I)\rho_{x12}\sigma_{xI}\sigma_{x2} + \alpha_I \beta_I \sigma_{txI}^2 + (1 - \alpha_I)\beta_I \sigma_{txI}^2 + \sigma_{cxI}^2$$
$$= \sigma_{x I DGZ_I^{as}}^2 + \sigma_{cxI}^2$$

$$\sigma_{y I AGZ_I^s}^2 \equiv \mathrm{Var}[Y_{AGZ_I^s} - Y_I] = (\alpha_I \beta_I - 1)^2 \sigma_{yI}^2 + ((1 - \alpha_I)\beta_I)^2 \sigma_{y2}^2$$
$$+ 2(\alpha_I \beta_I - 1)((1 - \alpha_I)\beta_I)\rho_{y12}\sigma_{yI}\sigma_{y2} + \alpha_I \beta_I \sigma_{tyI}^2 + (1 - \alpha_I)\beta_I \sigma_{tyI}^2 + \sigma_{cyI}^2$$
$$= \sigma_{y I DGZ_I^{as}}^2 + \sigma_{cyI}^2$$

$$\mu_{I AGZ_I^s} \equiv \mu_{x I AGZ_I^s}^2 + \mu_{y I AGZ_I^s}^2 + \sigma_{x I AGZ_I^s}^2 + \sigma_{y I AGZ_I^s}^2$$

$$\sigma_{I AGZ_I^s}^2 \equiv 2\sigma_{x I AGZ_I^s}^2(\sigma_{x I AGZ_I^s}^2 + 2\mu_{x I AGZ_I^s}^2) + 2\sigma_{y I AGZ_I^s}^2(\sigma_{y I AGZ_I^s}^2 + 2\mu_{y I AGZ_I^s}^2)$$

Evaluating the terms, recalling that $\alpha_I = 0.598$ and $\beta_I = 1.164$,

$$\mu_{x I AGZ_I^s} = -164 + 0 = -164$$
$$\mu_{y I AGZ_I^s} = -164 + 0 = -164$$
$$\sigma_{x I AGZ_I^s}^2 = 680,629 + 225^2 = 731,254$$
$$\sigma_{y I AGZ_I^s}^2 = 9,248 + 225^2 = 59,873$$
$$\mu_{I AGZ_I^s} = 844,919$$
$$\sigma_{I AGZ_I^s}^2 = 1.162 \times 10^{12}; \quad \sigma_{I AGZ_I^s} = 1,073,845$$

The conditional probability of defeat given weapon type 1 and a DGZ shift is:

$$P_{defeat}(1 | w_I, a_{12}, s, agg_{12}) \quad = P[D_{I AGZ_I^s}^2 \leq d_I^2 w_I] = P\left[Z \leq \frac{925^2 - \mu_{I AGZ_I^s}}{\sigma_{I AGZ_I^s}}\right] = P[Z \leq 0.010]$$
$$= 0.504$$

(b) *Unit #2*:

$$X_{AGZ_I^s} - X_2$$
$$= \alpha_I \beta_I X_I + (\beta_I - \alpha_I \beta_I - 1) X_2 + \alpha_I \beta_I TX_I + (1 - \alpha_I)\beta_I TX_2 + (1 - \beta) x_{P_I} + CX_I$$
$$= X_{DGZ_I^{as}} - X_2 + CX_I$$

$$Y_{AGZ_1^s} - Y_2$$

$$= \alpha_1\beta_1\, Y_1 + (\beta_1-\alpha_1\beta_1-1)\, Y_2 + \alpha_1\beta_1\, TY_1 + (1-\alpha_1)\beta_1\, TY_2 + (1-\beta)\, y_{P_1} + CY_1$$

$$= Y_{DGZ_1^{as}} - Y_2 + CY_1$$

Let

$$\mu_{x2AGZ_1^s} \equiv \alpha_1\beta_1\,\mu_{x1} + (\beta_1-\alpha_1\beta_1-1)\,\mu_{x2} + \alpha_1\beta_1\,\mu_{tx1} + (1-\alpha_1)\beta_1\,\mu_{tx2} + (1-\beta_1)\,x_{P_1} + \mu_{cx1}$$

$$= \mu_{x2DGZ_1^{as}} + \mu_{cx1}$$

$$\mu_{y2AGZ_1^s} \equiv \alpha_1\beta_1\,\mu_{y1} + (\beta_1-\alpha_1\beta_1-1)\,\mu_{y2} + \alpha_1\beta_1\,\mu_{ty1} + (1-\alpha_1)\beta_1\,\mu_{ty2} + (1-\beta_1)\,y_{P_1} + \mu_{cy1}$$

$$= \mu_{y2DGZ_1^{as}} + \mu_{cy1}$$

$$\sigma^2_{x2AGZ_1^s} \equiv \mathrm{Var}[X_{AGZ_1^s}-X_2] = \beta_1^2\sigma^2_{x1} + (\beta_1-\alpha_1\beta_1-1)^2\sigma^2_{x2}$$

$$\quad + 2(\alpha_1\beta_1)(\beta_1-\alpha_1\beta_1-1)\rho_{x12}\sigma_{x1}\sigma_{x2} + \alpha_1\beta_1\sigma^2_{tx1} + (1-\alpha_1)\beta_1\sigma^2_{tx1} + \sigma^2_{cx1}$$

$$= \sigma^2_{x2DGZ_1^{as}} + \sigma^2_{cx1}$$

$$\sigma^2_{y2AGZ_1^s} \equiv \mathrm{Var}[Y_{AGZ_1^s}-Y_2] = \beta_1^2\sigma^2_{y1} + (\beta_1-\alpha_1\beta_1-1)^2\sigma^2_{y2}$$

$$\quad + 2(\alpha_1\beta_1)(\beta_1-\alpha_1\beta_1-1)\rho_{y12}\sigma_{y1}\sigma_{y2} + \alpha_1\beta_1\sigma^2_{ty1} + (1-\alpha_1)\beta_1\sigma^2_{ty1} + \sigma^2_{cy1}$$

$$= \sigma^2_{y2DGZ_1^{as}} + \sigma^2_{cy1}$$

$$\mu_{2AGZ_1^s} \equiv \mu^2_{x2AGZ_1^s} + \mu^2_{y2AGZ_1^s} + \sigma^2_{x2AGZ_1^s} + \sigma^2_{y2AGZ_1^s}$$

$$\sigma^2_{2AGZ_1^s} \equiv 2\sigma^2_{x2AGZ_1^s}(\sigma^2_{x2AGZ_1^s}+2\mu^2_{x2AGZ_1^s}) + 2\sigma^2_{y2AGZ_1^s}(\sigma^2_{y2AGZ_1^s}+2\mu^2_{y2AGZ_1^s})$$

Evaluating the terms, recalling that $\alpha_1 = 0.598$ and $\beta_1 = 1.164$,

$$\mu_{x2AGZ_1^s} = -164 + 0 = -164$$

$$\mu_{y2AGZ_1^s} = -750 + 0 = -750$$

$$\sigma^2_{x2AGZ_1^s} = 1{,}592{,}629 + 225^2 = 1{,}643{,}254$$

$$\sigma^2_{y2AGZ_1^s} = 11{,}712 + 225^2 = 62{,}337$$

$$\mu_{2AGZ_1^s} = 2{,}294{,}987$$

$$\sigma^2_{2AGZ_1^s} = 5.725 \times 10^{12}; \quad \sigma_{2AGZ_1^s} = 2{,}392{,}780$$

The conditional probability of defeat of unit #2 given weapon type 1 and a DGZ shift is:

$$P_{defeat}(2\,|\,w_1, a_{12}, s, agg_{12}) = P[\, D^2_{2AGZ_1^s} \le d^2_{2\,W_1}\,] = P\left[Z \le \frac{925^2 - \mu_{2AGZ_1^s}}{\sigma_{2AGZ_1^s}}\right] = P[\,Z \le -0.602\,]$$

$$= 0.274$$

(4) DGZ shift, Weapon type 2:

We have previously evaluated the distribution of the shifted DGZ for the aggregate target, with random coordinates $(X_{DGZ_2^{as}}, Y_{DGZ_2^{as}})$. The AGZ coordinates $(X_{AGZ_2^s}, Y_{AGZ_2^s})$ may simply be

determined by adding the random variables for the CEP shift in the X and Y directions (CX_2 and CY_2 respectively) to the X and Y coordinates of the DGZ. Thus

$$X_{AGZ_2^s} = X_{DGZ_2^{as}} + CX_2 \quad \text{and} \quad Y_{AGZ_2^s} = Y_{DGZ_2^{as}} + CY_2.$$

(a) *Unit #1:*

$X_{AGZ_2^s} - X_1$
$$= (\alpha_2\beta_2 - 1) X_1 + (1-\alpha_2)\beta_2 X_2 + \alpha_2\beta_2 TX_1 + (1-\alpha_2)\beta_2 TX_2 + (1-\beta) x_{P_1} + CX_1$$

$Y_{AGZ_2^s} - Y_1$
$$= (\alpha_2\beta_2 - 1) Y_1 + (1-\alpha_2)\beta_2 Y_2 + \alpha_2\beta_2 TY_1 + (1-\alpha_2)\beta_2 TY_2 + (1-\beta) y_{P_1} + CY_1$$

Let

$$\mu_{x1AGZ_2^s} \equiv (\alpha_2\beta_2 - 1)\mu_{x1} + (1-\alpha_2)\beta_2\mu_{x2} + \alpha_2\beta_2\mu_{tx1} + (1-\alpha_2)\beta_2\mu_{tx2} + (1-\beta_2)x_{P_1} + \mu_{cx1}$$

$$= \mu_{x1DGZ_2^{as}} + \mu_{cx2}$$

$$\mu_{y1AGZ_2^s} \equiv (\alpha_2\beta_2 - 1)\mu_{y1} + (1-\alpha_2)\beta_2\mu_{y2} + \alpha_2\beta_2\mu_{ty1} + (1-\alpha_2)\beta_2\mu_{ty2} + (1-\beta_2)y_{P_1} + \mu_{cy1}$$

$$= \mu_{y1DGZ_2^{as}} + \mu_{cy2}$$

$$\sigma^2_{x1AGZ_2^s} \equiv \text{Var}[X_{AGZ_2^s} - X_1] = (\alpha_2\beta_2 - 1)^2\sigma^2_{x1} + ((1-\alpha_2)\beta_2)^2\sigma^2_{x2}$$
$$+ 2(\alpha_2\beta_2 - 1)((1-\alpha_2)\beta_2)\rho_{x12}\sigma_{x1}\sigma_{x2} + \alpha_2\beta_2\sigma^2_{tx1} + (1-\alpha_2)\beta_2\sigma^2_{tx1} + \sigma^2_{cx1}$$

$$= \sigma^2_{x1DGZ_2^{as}} + \sigma^2_{cx2}$$

$$\sigma^2_{y1AGZ_2^s} \equiv \text{Var}[Y_{AGZ_2^s} - Y_1] = (\alpha_2\beta_2 - 1)^2\sigma^2_{y1} + ((1-\alpha_2)\beta_2)^2\sigma^2_{y2}$$
$$+ 2(\alpha_2\beta_2 - 1)((1-\alpha_2)\beta_2)\rho_{y12}\sigma_{y1}\sigma_{y2} + \alpha_2\beta_2\sigma^2_{ty1} + (1-\alpha_2)\beta_2\sigma^2_{ty1} + \sigma^2_{cy1}$$

$$= \sigma^2_{y1DGZ_2^{as}} + \sigma^2_{cy2}$$

$$\mu_{1AGZ_2^s} \equiv \mu^2_{x1AGZ_2^s} + \mu^2_{y1AGZ_2^s} + \sigma^2_{x1AGZ_2^s} + \sigma^2_{y1AGZ_2^s}$$

$$\sigma^2_{1AGZ_2^s} \equiv 2\sigma^2_{x1AGZ_2^s}(\sigma^2_{x1AGZ_2^s} + 2\mu^2_{x1AGZ_2^s}) + 2\sigma^2_{y1AGZ_2^s}(\sigma^2_{y1AGZ_2^s} + 2\mu^2_{y1AGZ_2^s})$$

Evaluating the terms, recalling that $\alpha_2 = 0.594$ and $\beta_2 = 1.272$,

$$\mu_{x1AGZ_2^s} = -272 + 0 = -272$$

$$\mu_{y1AGZ_2^s} = 295.6 + 0 = 295.6$$

$$\sigma^2_{x1AGZ_2^s} = 804{,}265 + 225^2 = 854{,}890$$

$$\sigma^2_{y1AGZ_2^s} = 12{,}345 + 225^2 = 62{,}970$$

$$\mu_{1AGZ_2^s} = 992{,}140$$

$$\sigma^2_{1AGZ_2^s} = 1.723 \times 10^{12}; \quad \sigma_{1AGZ_2^s} = 1{,}312{,}505$$

The conditional probability of defeat given weapon type 2 and a DGZ shift is:

$$P_{defeat}(1 \mid w_2, a_{12}, s, agg_{12}) = P[\, D^2_{I\,AGZ^s_2} \leq d^2_{I\,W_2}\,] = P[\, Z \leq \frac{1355^2 - \mu_{I\,AGZ^s_2}}{\sigma_{I\,AGZ^s_2}}\,]$$
$$= P[\, Z \leq 0.643\,] = 0.740.$$

(b) *Unit #2:*

$$X_{AGZ^s_2} - X_2$$
$$= \alpha_2\beta_2\, X_I + (\beta_2 - \alpha_2\beta_2 - 1)\, X_2 + \alpha_2\beta_2\, TX_I + (1-\alpha_2)\beta_2\, TX_2 + (1-\beta)\, x_{P_I} + CX_I$$
$$Y_{AGZ^s_2} - Y_2$$
$$= \alpha_2\beta_2\, Y_I + (\beta_2 - \alpha_2\beta_2 - 1)\, Y_2 + \alpha_2\beta_2\, TY_I + (1-\alpha_2)\beta_2\, TY_2 + (1-\beta)\, y_{P_I} + CY_I$$

Let

$$\mu_{x2AGZ^s_2} \equiv \alpha_2\beta_2\,\mu_{xI} + (\beta_2 - \alpha_2\beta_2 - 1)\,\mu_{x2} + \alpha_2\beta_2\,\mu_{txI} + (1-\alpha_2)\beta_2\,\mu_{tx2} + (1-\beta_2)\,x_{P_I} + \mu_{cxI}$$
$$= \mu_{x2DGZ^{as}_2} + \mu_{cx2}$$
$$\mu_{y2AGZ^s_2} \equiv \alpha_2\beta_2\,\mu_{yI} + (\beta_2 - \alpha_2\beta_2 - 1)\,\mu_{y2} + \alpha_2\beta_2\,\mu_{tyI} + (1-\alpha_2)\beta_2\,\mu_{ty2} + (1-\beta_2)\,y_{P_I} + \mu_{cyI}$$
$$= \mu_{y2DGZ^{as}_2} + \mu_{cy2}$$
$$\sigma^2_{x2AGZ^s_2} \equiv \mathrm{Var}[X_{AGZ^s_2} - X_2] = (\alpha_2\beta_2)^2\sigma^2_{xI} + (\beta_2 - \alpha_2\beta_2 - 1)^2\sigma^2_{x2}$$
$$+ 2(\alpha_2\beta_2)(\beta_2 - \alpha_2\beta_2 - 1)\rho_{x12}\sigma_{xI}\sigma_{x2} + \alpha_2\beta_2\sigma^2_{txI} + (1-\alpha_2)\beta_2\sigma^2_{tx1} + \sigma^2_{cxI}$$
$$= \sigma^2_{x2DGZ^{as}_2} + \sigma^2_{cx2}$$
$$\sigma^2_{y2AGZ^s_2} \equiv \mathrm{Var}[Y_{AGZ^s_2} - Y_2] = (\alpha_2\beta_2)^2\sigma^2_{yI} + (\beta_2 - \alpha_2\beta_2 - 1)^2\sigma^2_{y2}$$
$$+ 2(\alpha_2\beta_2)(\beta_2 - \alpha_2\beta_2 - 1)\rho_{y12}\sigma_{yI}\sigma_{y2} + \alpha_2\beta_2\sigma^2_{tyI} + (1-\alpha_2)\beta_2\sigma^2_{ty1} + \sigma^2_{cyI}$$
$$= \sigma^2_{y2DGZ^{as}_2} + \sigma^2_{cy2}$$
$$\mu_{2AGZ^s_2} \equiv \mu^2_{x2AGZ^s_2} + \mu^2_{y2AGZ^s_2} + \sigma^2_{x2AGZ^s_2} + \sigma^2_{y2AGZ^s_2}$$
$$\sigma^2_{2AGZ^s_2} \equiv 2\sigma^2_{x2AGZ^s_2}(\sigma^2_{x2AGZ^s_2} + 2\mu^2_{x2AGZ^s_2}) + 2\sigma^2_{y2AGZ^s_2}(\sigma^2_{y2AGZ^s_2} + 2\mu^2_{y2AGZ^s_2})$$

Evaluating the terms, recalling that $\alpha_2 = 0.594$ and $\beta_2 = 1.272$,

$$\mu_{x2AGZ^s_2} = -272 + 0 = -272$$
$$\mu_{y2AGZ^s_2} = -804.4 + 0 = -804.4$$
$$\sigma^2_{x2AGZ^s_2} = 1{,}764{,}265 + 225^2 = 1{,}814{,}890$$
$$\sigma^2_{y2AGZ^s_2} = 14{,}264 + 225^2 = 64{,}889$$
$$\mu_{2AGZ^s_2} = 2{,}600{,}822$$
$$\sigma^2_{2AGZ^s_2} = 7.301 \times 10^{12}; \quad \sigma_{2AGZ^s_2} = 2{,}702{,}060$$

The conditional probability of defeat of unit #2 given weapon type 1 and a DGZ shift is:

$$P_{defeat}(2 \mid w_2, a_{12}, s, agg_{12}) \quad = P[\, D^2_{2AGZ_2^s} \leq d^2_{2W_1} \,] = P[\, Z \leq \frac{1355^2 - \mu_{2AGZ_2^s}}{\sigma_{2AGZ_2^s}} \,]$$
$$= P[\, Z \leq -0.283\,] = 0.389.$$

Computing the probabilities of engagement of the aggregate target

$$P_{engage}(\,1,2;\, \overline{s} \mid w_1\,) \quad = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\ shift}(1,2 \mid w_1, a_{12}, agg_{12}) \cdot p_{aggr}(1,2 \mid w_1, a_{12})$$
$$= (1)(1)(0.353)(0.480) = 0.169$$

$$P_{engage}(\,1,2;\, s \mid w_1\,) \quad = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(1,2 \mid w_1, a_{12}, agg_{12}) \cdot p_{aggr}(1,2 \mid w_1, a_{12})$$
$$= (1)(1)(0.093)(0.480) = 0.045$$

$$P_{engage}(\,1,2;\, \overline{s} \mid w_2\,) \quad = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\ shift}(1,2 \mid w_2, a_{12}, agg_{12}) \cdot p_{aggr}(1,2 \mid w_2, a_{12})$$
$$= (1)(1)(0.147)(0.616) = 0.091$$

$$P_{engage}(\,1,2;\, s \mid w_2\,) \quad = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(1,2 \mid w_2, a_{12}, agg_{12}) \cdot p_{aggr}(1,2 \mid w_2, a_{12})$$
$$= (1)(1)(0.229)(0.616) = 0.141$$

Suppose that $P_{round}(w_1 \mid a_{12}) = 0.4$ and $P_{round}(w_2 \mid a_{12}) = 0.6$. Then

$$P_{engage}(\,1,2;\, \overline{s};\, w_1\,) = P_{engage}(\,1,2;\, \overline{s} \mid w_1\,) \cdot P_{round}(w_1 \mid a_{12}) = 0.169\,(0.4) = 0.068$$
$$P_{engage}(\,1,2;\, s;\, w_1\,) = P_{engage}(\,1,2;\, s \mid w_1\,) \cdot P_{round}(w_1 \mid a_{12}) = 0.045\,(0.4) = 0.018$$
$$P_{engage}(\,1,2;\, \overline{s};\, w_2\,) = P_{engage}(\,1,2;\, \overline{s} \mid w_2\,) \cdot P_{round}(w_2 \mid a_{12}) = 0.091\,(0.6) = 0.055$$
$$P_{engage}(\,1,2;\, s;\, w_2\,) = P_{engage}(\,1,2;\, s \mid w_2\,) \cdot P_{round}(w_2 \mid a_{12}) = 0.141\,(0.6) = 0.085$$

SECTION V. COMPUTING THE PROBABILITIE, OF DEFEAT

Computing the probabilities of defeat of target unit #1

$$P_{defeat}(\,1 \mid w_1\,) = P_{defeat}(1 \mid w_1, a_1, \overline{s}, \overline{agg}_1) \cdot P_{engage}(\,1, \overline{s} \mid w_1\,)$$
$$+ P_{defeat}(1 \mid w_1, a_1, s, \overline{agg}_1) \cdot P_{engage}(\,1, s \mid w_1\,)$$
$$+ P_{defeat}(1 \mid w_1, a_{12}, \overline{s}, agg_{12}) \cdot P_{engage}(\,1,2;\, \overline{s} \mid w_1\,)$$
$$+ P_{defeat}(1 \mid w_1, a_{12}, s, agg_{12}) \cdot P_{engage}(\,1,2;\, s \mid w_1\,)$$
$$= 1.0\,(0.309) + 0.744\,(0.161) + 0.462\,(0.169) + 0.504\,(0.045) = 0.530$$

$$p_{defeat}(1 \mid w_2) = p_{defeat}(1 \mid w_2, a_1, \overline{s}, a\overline{g}g_1) \cdot p_{engage}(1, \overline{s} \mid w_2)$$

$$+ \; p_{defeat}(1 \mid w_2, a_1, s, a\overline{g}g_1) \cdot p_{engage}(1, s \mid w_2)$$

$$+ \; p_{defeat}(1 \mid w_2, a_{12}, \overline{s}, agg_{12}) \cdot p_{engage}(1,2; \overline{s} \mid w_2)$$

$$+ \; p_{defeat}(1 \mid w_2, a_{12}, s, agg_{12}) \cdot p_{engage}(1,2; s \mid w_2)$$

$$= 1.0\,(0.180) + 0.738\,(0.233) + 0.799\,(0.091) + 0.740\,(0.141) = 0.529$$

$$p_{defeat}(1, w_1) = p_{defeat}(1 \mid w_1, a_1, \overline{s}, a\overline{g}g_1) \cdot p_{engage}(1, \overline{s}, w_1)$$

$$+ \; p_{defeat}(1 \mid w_1, a_1, s, a\overline{g}g_1) \cdot p_{engage}(1, s, w_1)$$

$$+ \; p_{defeat}(1 \mid w_1, a_{12}, \overline{s}, agg_{12}) \cdot p_{engage}(1,2; \overline{s}, w_1)$$

$$+ \; p_{defeat}(1 \mid w_1, a_{12}, s, agg_{12}) \cdot p_{engage}(1,2; s, w_1)$$

$$= 1.0\,(0.232) + 0.744\,(0.121) + 0.462\,(0.068) + 0.504\,(0.018) = 0.363$$

$$p_{defeat}(1, w_2) = p_{defeat}(1 \mid w_2, a_1, \overline{s}, a\overline{g}g_1) \cdot p_{engage}(1, \overline{s}, w_2)$$

$$+ \; p_{defeat}(1 \mid w_2, a_1, s, a\overline{g}g_1) \cdot p_{engage}(1, s, w_2)$$

$$+ \; p_{defeat}(1 \mid w_2, a_{12}, \overline{s}, agg_{12}) \cdot p_{engage}(1,2; \overline{s}, w_2)$$

$$+ \; p_{defeat}(1 \mid w_2, a_{12}, s, agg_{12}) \cdot p_{engage}(1,2; s, w_2)$$

$$= 1.0\,(0.045) + 0.738\,(0.058) + 0.799\,(0.055) + 0.740\,(0.085) = 0.195$$

With only two weapon types available in this example,

$$p_{defeat}(1) \quad = p_{defeat}(1, w_1) + p_{defeat}(1, w_2)$$

$$= 0.363 + 0.195 = 0.558$$

Computing the probabilities of defeat of target unit #2

$$p_{defeat}(2 \mid w_1) = p_{defeat}(2 \mid w_1, a_2, \overline{s}, a\overline{g}g_2) \cdot p_{engage}(2, \overline{s} \mid w_1)$$

$$+ \; p_{defeat}(2 \mid w_1, a_2, s, a\overline{g}g_2) \cdot p_{engage}(2, s \mid w_1)$$

$$+ \; p_{defeat}(2 \mid w_1, a_{12}, \overline{s}, agg_{12}) \cdot p_{engage}(1,2; \overline{s} \mid w_1)$$

$$+ \; p_{defeat}(2 \mid w_1, a_{12}, s, agg_{12}) \cdot p_{engage}(1,2; s \mid w_1)$$

$$= 1.0\,(0.260) + 1.0\,(0.158) + 0.299\,(0.169) + 0.274\,(0.045) = 0.481$$

$$p_{defeat}(2 \mid w_2) = p_{defeat}(2 \mid w_2, a_2, \overline{s}, a\overline{g}g_2) \cdot p_{engage}(2, \overline{s} \mid w_2)$$

$$+ \; p_{defeat}(2 \mid w_2, a_2, s, a\overline{g}g_2) \cdot p_{engage}(2, s \mid w_2)$$

$$+ \; p_{defeat}(2 \mid w_2, a_{12}, \overline{s}, agg_{12}) \cdot p_{engage}(1,2; \overline{s} \mid w_2)$$

$$+ \; p_{defeat}(2 \mid w_2, a_{12}, s, agg_{12}) \cdot p_{engage}(1,2; s \mid w_2)$$

$$= 1.0\,(0.142) + 0.981\,(0.217) + 0.727\,(0.091) + 0.389\,(0.141) = 0.476$$

$$p_{defeat}(2, w_1) = p_{defeat}(2 \mid w_1, a_2, \overline{s}, a\overline{g}g_2) \cdot p_{engage}(2, \overline{s}, w_1)$$

$$+ \; p_{defeat}(2 \mid w_1, a_2, s, a\overline{g}g_2) \cdot p_{engage}(2, s, w_1)$$

$$+ \; p_{defeat}(2 \mid w_1, a_{12}, \overline{s}, agg_{12}) \cdot p_{engage}(1,2; \overline{s}, w_1)$$

$$+ \; p_{defeat}(2 \mid w_1, a_{12}, s, agg_{12}) \cdot p_{engage}(1,2; s, w_1)$$

$$= 1.0(0.065) + 1.0(0.040) + 0.299(0.068) + 0.274(0.018) = 0.130$$

$$p_{defeat}(2, w_2) = p_{defeat}(2 \mid w_2, a_2, \overline{s}, a\overline{g}g_2) \cdot p_{engage}(2, \overline{s}, w_2)$$

$$+ \; p_{defeat}(2 \mid w_2, a_2, s, a\overline{g}g_2) \cdot p_{engage}(2, s, w_2)$$

$$+ \; p_{defeat}(2 \mid w_2, a_{12}, \overline{s}, agg_{12}) \cdot p_{engage}(1,2; \overline{s}, w_2)$$

$$+ \; p_{defeat}(2 \mid w_2, a_{12}, s, agg_{12}) \cdot p_{engage}(1,2; s, w_2)$$

$$= 1.0(0.107) + 0.981(0.163) + 0.727(0.055) + 0.389(0.085) = 0.340$$

With only two weapon types available in this example,

$$p_{defeat}(2) = p_{defeat}(2, w_1) + p_{defeat}(2, w_2)$$

$$= 0.130 + 0.340 = 0.470$$

APPENDIX H
EXAMPLE OF P_{ROUND} PARAMETER ESTIMATION

Section I - COMPUTING THE RELEVANT PROBABILITIES

Given Information

Suppose there are 10 targetable units with the following characteristics:

Unit # 1,2,3 : Infantry battalion   Priority : 3

Unit # 4,5,6,7 : Tank company   Priority : 2

Unit # 8,9 : Artillery battery   Priority : 2

Unit # 10 : Missile launcher   Priority : 1

Suppose there are 3 firing units available to attack these units:

Firing Unit # 1,2 : Artillery battery   Weapon : 1 kt AFAP

Firing Unit # 3 : Missile launcher   Weapon : 10 kt missile

We have the following range information: For each firing unit, the following targets are in range:

Firing Unit # 1 : Units # 1,2,4,5,8

Firing Unit # 2 : Units # 1,2,3,5,6,8,9

Firing Unit # 3 : All Units

We have the following fire preferences:

Unit #	First preference	Second preference
1,2,3 : Infantry bn.	10 kt missile	None (1 kt too small)
4,5,6,7 : Tank co.	1 kt AFAP	10 kt missile
8,9 : Artillery bt.	1 kt AFAP	10 kt missile
10 : Missile ln.	10 kt missile	None (all arty out of range)

If we combine the fire preference with the range information, we can tabulate a binary variable $RF(i,j,w) = 1$ if unit i can be fired on by firing unit j using weapon w; 0 otherwise:

Unit	1	2	3	4	5	6	7	8	9	10
FU #1	1	1	0	1	1	0	0	1	0	0
FU #2	1	1	1	0	1	1	0	1	1	0
FU #3	0	0	0	1	1	1	1	1	1	1

NOTE that the smaller yield weapon is always preferred in cases where either weapon is available ("bottom-up" allocation).

Computing the Probabilities

(1) *Weapon # 1 (1 kt) :*

We are given the following values for $p_{aggr}(ij|\ w_1, a_{ij})$:

Unit	1	2	3	4	5	6	7	8	9	10
$j = 1$	-	.4	.1	.2	0	0	0	.1	0	0
$j = 2$	.4	-	.6	.1	.3	.1	0	.1	.2	0
$j = 3$	.1	.6	-	.5	.2	0	.1	0	.3	0
$j = 4$	.2	.1	.5	-	.4	.2	.1	0	0	0
$j = 5$	0	.3	.2	.4	-	.4	.2	.1	0	0
$j = 6$	0	.1	0	.2	.4	-	.4	0	.2	0
$j = 7$	0	0	.1	.1	.2	.4	-	.4	.1	0
$j = 8$	.1	.1	0	0	.1	0	.4	-	.1	0
$j = 9$	0	.2	.3	0	0	.2	.1	.1	-	0
$j = 10$	0	0	0	0	0	0	0	0	0	-

We are given the following values for $p_{avail}(i)$ and $\overline{p}_{prec}(i|\ w_1, a_i, a\overline{g}g_i)$:

Unit	1	2	3	4	5	6	7	8	9	10
$p_{avail}(i)$	.5	.6	.5	.7	.7	.7	.5	.4	.3	.2
$\overline{p}_{prec}(i|$	.3	.6	.9	.4	.6	.8	1.0	.4	.9	1.0

From $p_{aggr}(ij|\ w_1, a_{ij})$, $p_{avail}(i)$, and $\overline{p}_{prec}(i|\ w_1, a_i, a\overline{g}g_i)$ we can compute $\overline{p}_{aggr}(i|\ w_1, a_{12})$. Recall that $\overline{p}_{aggr}(i|\ w_1, a_i) = 1 - \max_j \left\{ p_{aggr}(ij|\ w_1, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j|\ w_1, a_j, a\overline{g}g_j) \right\}$. The following table shows values for the ith column, jth row of $p_{aggr}(ij|\ w_1, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j|\ w_1, a*_j, a\overline{g}g_j)$. For example, the calculation for $i = 2$, $j = 4$: $p_{aggr}(ij|\ w_1, a_{ij}) = .1$; $p_{avail}(j) = .7$; $\overline{p}_{prec}(j|\ w_1, a_j, a\overline{g}g_j) = .4$; product = .028.

$i =$	1	2	3	4	5	6	7	8	9	10
$j = 1$	-	.060	.015	.030	0	0	0	.015	0	0
$j = 2$	.144	-	.216	.036	.108	.036	0	.036	.072	0
$j = 3$	.045	.270	-	.225	.090	0	.045	0	.135	0
$j = 4$	.036	.028	.140	-	.112	.056	.028	0	0	0
$j = 5$	0	.126	.084	.168	-	.168	.084	.043	0	0
$j = 6$	0	.056	0	.112	.224	-	.224	0	.112	0
$j = 7$	0	0	.050	.050	.100	.200	-	.200	.050	0
$j = 8$	.016	.016	0	0	.016	0	.064	-	.016	0
$j = 9$	0	.054	.081	0	0	.027	.027	.027	-	0
$j = 10$	0	0	0	0	0	0	0	0	0	-

Choosing the largest value in each column gives us $\max_j \left\{ p_{aggr}(ij| \ w_I, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j| \ w_I, a_j, a\overline{g}g_j) \right\}$ with the corresponding j^* that maximizes the product. Thus $\overline{p}_{aggr}(i| \ w_I, a_i)$ values for $i = 1, \ldots, 10$ are:

Unit	1	2	3	4	5	6	7	8	9	10
$\overline{p}_{aggr}(i|$	.856	.730	.784	.775	.776	.800	.776	.800	.865	1.0
j^*	2	3	2	3	6	7	6	7	3	-

We are given the following values for $\overline{p}_{prec}(i| \ w_I, a_i, a\overline{g}g_i)$, $p_{no\ shift}(i| \ w_I, a_i, a\overline{g}g_i)$, and $p_{shift}(i| \ w_I, a_i, a\overline{g}g_i)$:

Unit	1	2	3	4	5	6	7	8	9	10
$\overline{p}_{prec}(i|$	.3	.6	.9	.4	.6	.8	1.0	.4	.9	1.0
$p_{no\ shift}(i|$	.2	.4	.6	.2	.4	.5	.9	.3	.5	1.0
$p_{shift}(i|$	.1	.2	.3	.2	.2	.3	.1	.1	.4	0

We are given the following values for $\overline{p}_{prec}(ij| w_I, a_{ij}, agg_{ij})$, $p_{no\ shift}(ij| w_I, a_{ij}, agg_{ij})$, and $p_{shift}(ij| \ w_I, a_{ij}, agg_{ij})$ for the j^* values which maximize $\max_j \left\{ p_{aggr}(ij| \ w_I, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j| \ w_I, a_j, a\overline{g}g_j) \right\}$.

Pair	1,2	2,3	3,2	4,3	5,6	6,7	7,6	8,7	9,3	10
$\overline{p}_{prec}(ij|$	.5	.7	.7	.7	.7	.9	.9	.6	.9	-
$p_{no\ shift}(ij|$	.2	.4	.4	.35	.4	.5	.5	.4	.6	-
$p_{shift}(ij|$	.2	.2	.2	.3	.25	.35	.35	.15	.25	-

(2) *Weapon # 2 (10 kt) :*

We are given the following values for $p_{aggr}(ij|\ w_2, a_{ij})$:

Unit	1	2	3	4	5	6	7	8	9	10
$j = 1$	-	.6	.3	.4	.1	0	.1	..4	.1	0
$j = 2$	.6	-	..8	.2	.5	.3	.1	.2	.3	0
$j = 3$	.3	.8	-	.9	.5	.1	.3	.1	.5	0
$j = 4$	.4	.2	.9	-	.8	.4	.2	.2	.1	0
$j = 5$	.1	.5	.5	.8	-	.9	.3	.2	.2	0
$j = 6$	0	.3	.1	.4	.9	-	.8	.1	.4	0
$j = 7$	.1	.1	.3	.2	.3	.8	-	.8	.3	0
$j = 8$	.4	.2	.1	.2	..2	.1	.8	-	.2	0
$j = 9$	.1	.3	.5	.1	.2	.4	.3	.2	-	0
$j = 10$	0	0	0	0	0	0	0	0	0	-

We are given the following values for $p_{avail}(i)$ and $\overline{p}_{prec}(i|\ w_2, a_i, a\overline{g}g_i)$:

Unit	1	2	3	4	5	6	7	8	9	10
$p_{avail}(i)$	.5	.6	.5	.7	.7	.7	.5	.4	.3	.2
$\overline{p}_{prec}(i|$	.2	.4	.8	.3	.5	.7	.9	.2	.7	1.0

From $p_{aggr}(ij|\ w_2, a_{ij})$, $p_{avail}(i)$, and $\overline{p}_{prec}(i|\ w_2, a_i, a\overline{g}g_i)$ we can compute $\overline{p}_{aggr}(i|\ w_2, a_{12})$. Recall that $\overline{p}_{aggr}(i|\ w_2, a_i) = 1 - \max_j \left\{ p_{aggr}(ij|\ w_2, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j|\ w_2, a_j, a\overline{g}g_j) \right\}$. The following table shows values for the ith column, jth row of $p_{aggr}(ij|\ w_2, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j|\ w_2, a_j, a\overline{g}g_j)$:

$i =$	1	2	3	4	5	6	7	8	9	10
$j = 1$	-	.060	.030	.040	.010	0	.010	.040	.010	0
$j = 2$	.144	-	.192	.048	.120	.072	.024	.048	.072	0
$j = 3$	.120	.320	-	.360	.200	.040	.120	.040	.200	0
$j = 4$	.084	.042	.189	-	.168	.084	.042	.042	.021	0
$j = 5$	.035	.175	.175	.280	-	.315	.105	.070	.070	0
$j = 6$	0	.147	.049	.196	.441	-	.392	.049	.196	0
$j = 7$	.045	.045	.135	.090	.135	.360	-	.360	.135	0
$j = 8$	.032	.016	.008	.016	.016	.008	.064	-	.016	0
$j = 9$	.021	.063	.105	.021	.042	.084	.063	.042	-	0
$j = 10$	0	0	0	0	0	0	0	0	0	-

Choosing the largest value in each column gives us $\max_j \left\{ p_{aggr}(ij \mid w_2, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j \mid w_2, a_j, a\overline{g}g_j) \right\}$. Thus $\overline{p}_{aggr}(i \mid w_2, a_i)$ values for $i = 1, \ldots, 10$ are:

Unit	1	2	3	4	5	6	7	8	9	10
$\overline{p}_{aggr}(i \mid$	.856	.680	.808	.640	.559	.640	.608	.640	.804	1.0
j^*	2	3	2	3	6	7	6	7	6	-

We are given the following values for $\overline{p}_{prec}(i \mid w_2, a_i, a\overline{g}g_i)$, $\overline{p}_{no\ shift}(i \mid w_2, a_i, a\overline{g}g_i)$, and $\overline{p}_{shift}(i \mid w_2, a_i, a\overline{g}g_i)$:

Unit	1	2	3	4	5	6	7	8	9	10
$\overline{p}_{prec}(i \mid$	.2	.4	.8	.3	.5	.7	.9	.2	.7	1.0
$\overline{p}_{no\ shift}(i \mid$	.1	.2	.3	.1	.2	.2	.3	.4	.1	.9
$\overline{p}_{shift}(i \mid$	.1	.2	.5	.2	.3	.4	.5	.1	.5	.1

We are given the following values for $\overline{p}_{prec}(ij \mid w_1, a_{ij}, agg_{ij})$, $\overline{p}_{no\ shift}(ij \mid w_1, a_{ij}, agg_{ij})$, and $\overline{p}_{shift}(ij \mid w_1, a_{ij}, agg_{ij})$ for the j^* values which maximize $\max_j \left\{ p_{aggr}(ij \mid w_1, a_{ij}) \cdot p_{avail}(j) \cdot \overline{p}_{prec}(j \mid w_1, a_j, a\overline{g}g_j) \right\}$.

Pair	1,2	2,3	3,2	4,3	5,6	6,7	7,6	8,7	9,6	10
$\overline{p}_{prec}(ij\|$	.3	.6	.6	.5	.6	.8	.8	.5	.7	-
$\overline{p}_{no\ shift}(ij\|$	.05	.25	.25	.2	.25	.35	.35	.2	.2	-
$\overline{p}_{shift}(ij\|$	.2	.3	.3	.25	.3	.4	.4	.25	.4	-

Computing the Conditional Probabilities of Engagement

Recall that $p_{engage}(i,\overline{s},\mid w) = p_{avail}(i) \cdot p_{no\ shift}(i\mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i\mid w, a_i)$ and
$p_{engage}(i,s,\mid w) = p_{avail}(i) \cdot p_{shift}(i\mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i\mid w, a_i)$.
Similarly, $p_{engage}(ij,\overline{s},\mid w) = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\ shift}(ij\mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij\mid w, a_{ij})$ and
$p_{engage}(ij,s,\mid w) = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(ij\mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij\mid w, a_{ij})$

(1) *Weapon # 1 (1 kt) :*

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(i,\overline{s}$	.086	.175	.235	.109	.217	.280	.349	.096	.130	.200
$p_{engage}(i,s$	.043	.088	.118	.109	.109	.168	.039	.032	.104	0
$p_{engage}(i\mid w$	.129	.263	.353	.218	.326	.448	.388	.128	.234	.200

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(ij,\overline{s}$	.024	.072	.072	.061	.176	.070	.070	.032	.027	-
$p_{engage}(ij,s$	.024	.036	.036	.053	.110	.049	.049	.012	.011	-
$p_{engage}(ij\mid w$	.048	.108	.108	.114	.286	.119	.119	.044	.038	-

(2) *Weapon # 2 (10 kt) :*

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(i,\overline{s}$	.040	.082	.121	.045	.078	.134	.122	.026	.048	.180
$p_{engage}(i,s$	.040	.082	.202	.090	.117	.179	.152	.026	.121	.020
$p_{engage}(i\mid w$	.080	.164	.323	.135	.195	.313	.274	.052	.169	.200

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(ij, \overline{s})$	.009	.060	.060	.063	.110	.098	.098	.032	.015	-
$p_{engage}(ij, s)$	.036	.072	.072	.079	.132	.112	.112	.040	.030	-
$p_{engage}(ij \mid w)$	.045	.132	.132	.142	.242	.210	.210	.072	.045	-

If we combine the single and aggregate probabilities to get a probability that target unit i is engaged *either* as a single or aggregate target, we get:

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(i_s \cup i_a \mid w_1)$	.177	.371	.461	.332	.440	.567	.507	.172	.272	.200
$p_{engage}(i_s \cup i_a \mid w_2)$	.125	.296	.455	.270	.437	.523	.484	.124	.214	.200

Section II - ESTABLISHING THE ALGORITHM FOR PARAMETER ESTIMATION

Theory

To develop a Monte Carlo estimate of the probability (by weapon type) that a round is available for a given target unit, we begin by generating realizations of single and aggregate target sets. To do this, we draw against the probability that the target unit is available for fire as follows:

ALGORITHM:

1. For each target unit i, $i = 1, ..., m$,

2.   Draw $U_i \sim$ Uniform(0,1).

3.   For each weapon type w, $w = 1, ..., nw$,

4.   Using j maximizing $\left\{ \overline{p}_{aggr}(ij \mid w, a_{ij}) \cdot p_{avail}(j \mid w) \cdot \overline{p}_{prec}(j \mid w, a_j, a\overline{g}g_j) \right\}$,
     if $U_i < p_{engage}(i \mid w) + p_{engage}(ij \mid w)$, let $B(i, w) = 1$

5.   Also if $U_i < p_{engage}(i \mid w)$, let $A(i, w) = 1$

6.   End if

7.   End loop on w

8. End loop on i

9. The available target set is generated as follows:

   If $B(i, w) = 0$, target unit i is not available for fire

   If $A(i, w) = 1$, target unit i is available for fire as a single target

   If $A(i, w) = 0$ and $B(i, w) = 1$, target unit i is available for fire as an aggregate target

For convenience in notation, define the following values for the encoding of the algorithm:

$$\text{PSINGLE}(i, w) \equiv p_{engage}(i \mid w)$$

$$\text{PAGGR}(i, w) \equiv p_{engage}(i \mid w) + p_{engage}(ij \mid w)$$

Section III - A PROGRAM FOR MONTE CARLO P_{round} ESTIMATION

A simple SIMSCRIPT program was written to generate Monte Carlo estimates of $p_{round}(w \mid a_i)$ for the 10 target units, 3 firing units and two weapon types given in Section I. This program is *not* intended as an example of an efficient program for generating these estimates; rather it serves as an example of an implementation of the algorithm in Section II. Implementation of the NEMESIS research will entail writing a more efficient program, operating with the same logic, to generate the estimates.

The Example Logic

The various data developed in the previous sections were read into the code. The single targets were defined for $i = 1, \ldots, 10$ and the aggregate targets were defined for $i = 11, \ldots, 20$ where, for example, target 14 was the aggregate pair formed by target unit #4 and its pair (in this case, target unit #3). The fire preference matrix and the range factor matrix were combined to form a single matrix $\text{FPREF}(I, J)$ where I = target unit and J = firing unit. For aggregate targets, the fire preference was equal to the product of the preferences for both target pairs. Thus, an aggregate target pair could only be fired upon by a firing unit if *both* of the target units were on the firing preference list for the firing unit. The fire priority for aggregate targets was assigned as the maximum of the priority for either of the aggregated targets. In one case (target unit #9), the aggregate pair depends upon the weapon type involved. Rather than establishing a 2-dimensional priority (unit by weapon type), I used a priority equal to the maximum of the first unit and the minimum of either of the second (aggregated) units selected by weapon (see line 58 of the code).

The output value was $p_{round}(w \mid a_i)$ for single targets and $p_{round}(w \mid a_i)$ for (i,j) aggregate target pairs. This probability represents the probability that a round of type w is available for assignment to the target i or ij, given that neither unit i or j have been allocated a weapon previously and that the target has a firing unit of the appropriate type that can engage the target (the fire preference is equal to 1). It is developed separately for single vs aggregate targets and incorporates the range factor consideration.

Three runs were made for illustration. The first run had only 1 weapon available per firing unit. The second run had 2 weapons available per firing unit, and the third run had 4 weapons available for firing units 1 and 3 with 2 weapons for firing unit 2. The p_{round} values are higher than might be expected, since the PSINGLE and PAGGR values are so low (since they are low, there were often only one or two units that were available for engagement per replication, thus a unit available for engagement generally had a weapon available in each replication).

SIMSCRIPT Code used to Generate Example

```
1  PREAMBLE
2     NORMALLY MODE IS REAL
3  END

1  MAIN
2     DEFINE PSINGLE,PAGGR AS REAL, 2-DIMENSIONAL ARRAYS
3     DEFINE A,B,CA,CB,PROUND,CROUND,IPAIR,FPREF AS INTEGER, 2-DIMENSIONAL
          ARRAYS
4     DEFINE PRI,ENGAGE AS INTEGER, 1-DIMENSIONAL ARRAYS
5     DEFINE I,J,K,L,NW1,NW2,NW3,N,W AS INTEGER VARIABLES
6     RESERVE PSINGLE(*,*) AS 10 BY 2
7     RESERVE PAGGR(*,*) AS 10 BY 2
8     RESERVE A(*,*) AS 10 BY 2
9     RESERVE B(*,*) AS 10 BY 2
10    RESERVE CA(*,*) AS 10 BY 2
11    RESERVE CB(*,*) AS 10 BY 2
12    RESERVE PROUND(*,*) AS 20 BY 2
13    RESERVE CROUND(*,*) AS 20 BY 2
14    RESERVE IPAIR(*,*) AS 10 BY 2
15    RESERVE PRI(*) AS 20
16    RESERVE FPREF(*,*) AS 20 BY 3
17    RESERVE ENGAGE(*) AS 10
18    FOR W = 1 TO 2 DO
19      FOR I = 1 TO 10 DO
20        READ PSINGLE(I,W)
21      LOOP
22    LOOP
23    FOR W = 1 TO 2 DO
24      FOR I = 1 TO 10 DO
25        READ PAGGR(I,W)
26      LOOP
27    LOOP
28    FOR W = 1 TO 2 DO
29      PRINT 1 LINE WITH W THUS
   WEAPON # ** :
31      FOR I = 1 TO 10 DO
32        READ IPAIR(I,W)
33        PRINT 1 LINE WITH I,W,PSINGLE(I,W), IPAIR(I,W), I,W,PAGGR(I,W) THUS
   PSINGLE(**,*) = *.***  FOR J = **,  PAGGR(**,*) = *.***
35      LOOP
36    LOOP
```

```
37    FOR J = 1 TO 3 DO
38      FOR I = 1 TO 10 DO
39        READ FPREF(I,J)
40      LOOP
41    LOOP
42    FOR I = 1 TO 10 DO
43      FOR J = 1 TO 2 DO
44        LET FPREF(10+I,J) = FPREF(I,J) * FPREF(IPAIR(I,1),J)
45      LOOP
46      LET FPREF(10+I,3) = FPREF(I,3) * FPREF(IPAIR(I,2),3)
47      PRINT 1 LINE WITH I,FPREF(I,1), I,FPREF(I,2), I,FPREF(I,3) THUS
   FPREF(**,1) = **     FPREF(**,2) = **     FPREF(**,3) = **
49    LOOP
50    LET FPREF(20,3) = 0   "CORRECTS FOR FACT THAT WEAPON 10 HAS NO
        AGGREGATE PAIR
51    FOR I = 11 TO 20 DO
52      PRINT 1 LINE WITH I,IPAIR(I-10,1),FPREF(I,1), I,IPAIR(I-10,1),FPREF(I,2),
53        I,IPAIR(I-10,2),FPREF(I,3) THUS
   FPREF(**,**,1) = **     FPREF(**,**,2) = **     FPREF(**,**,3) = **
55    LOOP
56    FOR I = 1 TO 10 DO
57      READ PRI(I)
58      LET PRI(10+I) = MAX.F( PRI(I), MIN.F(PRI(IPAIR(I,1)),PRI(IPAIR(I,2))) )
59      PRINT 1 LINE WITH I,PRI(I),I+10,PRI(I+10) THUS
   PRI(**) = **     PRI(**) = **
61    LOOP
62    READ NW1
63    READ NW2
64    READ NW3
65    READ N
66    PRINT 4 LINES WITH NW1, NW2, NW3, N THUS
   WEAPONS AVAILABLE:
     1 KT, FIRE UNIT 1 : **     1 KT, FIRE UNIT 2 : **     10 KT, FIRE UNIT 3 : **
   NUMBER OF REPLICATIONS :  ******

71    LET NW1T = NW1
72    LET NW2T = NW2
73    LET NW3T = NW3
74    FOR I = 1 TO 10 DO
75      FOR W = 1 TO 2 DO
76        LET A(I,W) = 0
77        LET B(I,W) = 0
78        LET CA(I,W) = 0
79        LET CB(I,W) = 0
80      LOOP
81      LET ENGAGE(I) = 0
82    LOOP
83    FOR K = 1 TO N DO
84      FOR I = 1 TO 10 DO
85        FOR W = 1 TO 2 DO
86          LET U = RANDOM.F(1)
87          IF U < PSINGLE(I,W) + PAGGR(I,W)
88            LET B(I,W) = 1
```

```
89          IF U < PSINGLE(I,W)
90            LET CA(I,W) = CA(I,W) + 1
91            LET A(I,W) = 1
92          ELSE LET CB(I,W) = CB(I,W) + 1
93          ALWAYS
94        ALWAYS
95      LOOP
96    LOOP
97    FOR L = 1 TO 3 DO
98    "
99    " AGGREGATE TARGETS  I = 11,20
100   "
101     FOR I = 1 TO 10 DO
102     IF PRI(I+10) NE L GO TO NEXTI ALWAYS
103     IF A(I,1) = 0 AND B(I,1) > 0    "AGGREGATE TARGET
104       IF FPREF(I+10,1) = 1 AND ENGAGE(I) = 0
105         LET CROUND(I+10,1) = CROUND(I+10,1) + 1
106         IF NW1T > 0
107           LET PROUND(I+10,1) = PROUND(I+10,1) + 1
108           LET ENGAGE(I) = 1
109           LET ENGAGE( IPAIR(I,1) ) = 1
110           LET NW1T = NW1T - 1
111 "PRINT 1 LINE WITH K,L,I,A(I,1),B(I,1),NW1T,PROUND(I+10,1) THUS
112 "REP=**** PRI=* I=** W=1 A=*** B=*** NW1=*** PROUND=***
113         ALWAYS "NW1T
114         GO TO NEXTI
115       ELSE IF FPREF(I+10,2) = 1 AND ENGAGE(I) = 0
116         LET CROUND(I+10,1) = CROUND(I+10,1) + 1
117         IF NW2T > 0
118           LET PROUND(I+10,1) = PROUND(I+10,1) + 1
119           LET ENGAGE(I) = 1
120           LET ENGAGE( IPAIR(I,1) ) = 1
121           LET NW2T = NW2T - 1
122 "PRINT 1 LINE WITH K,L,I,A(I,1),B(I,1),NW2T,PROUND(I+10,1) THUS
123 "REP=**** PRI=* I=** W=1 A=*** B=*** NW2=*** PROUND=***
124         ALWAYS "NW2T
125         GO TO NEXTI
126       ALWAYS "FPREF1 OR 2
127       ALWAYS "A=0, B(I,1)=1
128     IF A(I,2) = 0 AND B(I,2) > 0 AND FPREF(I+10,3) = 1 AND ENGAGE(I) = 0
129         LET CROUND(I+10,2) = CROUND(I+10,2) + 1
130         IF NW3T > 0
131           LET PROUND(I+10,2) = PROUND(I+10,2) + 1
132           LET ENGAGE(I) = 1
133           LET ENGAGE( IPAIR(I,2) ) = 1
134           LET NW3T = NW3T - 1
135 "PRINT 1 LINE WITH K,L,I,A(I,2),B(I,2),NW3T,PROUND(I+10,2) THUS
136 "REP=**** PRI=* I=** W=2 A=*** B=*** NW3=*** PROUND=***
137         ALWAYS "NW3T
138         GO TO NEXTI
139       ALWAYS "A=0, B(I,2)=1 AND FPREF3
140     'NEXTI' LOOP "ON UNIT I AGGREGATE TARGET
141     "
```

```
142    "   SINGLE TARGETS  I = 1, 10
143    "
144     FOR I = 1 TO 10 DO
145      IF PRI(I) NE L GO TO NEXTII ALWAYS
146       IF A(I,1) = 1           "SINGLE TARGET
147        IF FPREF(I,1) = 1 AND ENGAGE(I) = 0
148           LET CROUND(I,1) = CROUND(I,1) + 1
149          IF NW1T > 0
150             LET PROUND(I,1) = PROUND(I,1) + 1
151             LET NW1T = NW1T - 1
152  "PRINT 1 LINE WITH K,L,I,A(I,1),B(I,1),NW1T,PROUND(I,1) THUS
153  "REP=****  PRI=*  I=**  W=1 A=*** B=*** NW1=*** PROUND=***
154          ALWAYS  "NW1T
155          GO TO NEXTII
156        ELSE IF FPREF(I,2) = 1 AND ENGAGE(I) = 0
157           LET CROUND(I,1) = CROUND(I,1) + 1
158          IF NW2T > 0
159             LET PROUND(I,1) = PROUND(I,1) + 1
160             LET NW2T = NW2T - 1
161  "PRINT 1 LINE WITH K,L,I,A(I,1),B(I,1),NW2T,PROUND(I,1) THUS
162  "REP=****  PRI=*  I=**  W=1 A=*** B=*** NW2=*** PROUND=***
163          ALWAYS  "NW2T
164          GO TO NEXTII
165        ALWAYS  "FPREF1 OR 2
166        ALWAYS  "A(I,1) = 1
167        IF A(I,2) = 1 AND FPREF(I,3) = 1 AND ENGAGE(I) = 0
168           LET CROUND(I,2) = CROUND(I,2) + 1
169          IF NW3T > 0
170             LET PROUND(I,2) = PROUND(I,2) + 1
171             LET NW3T = NW3T - 1
172  "PRINT 1 LINE WITH K,L,I,A(I,2),B(I,2),NW3T,PROUND(I,2) THUS
173  "REP=****  PRI=*  I=**  W=2 A=*** B=*** NW3=*** PROUND=***
174          ALWAYS  "NW3T
175          GO TO NEXTII
176        ALWAYS  "A(I,2) AND FPREF3
177     'NEXTII' LOOP  "ON UNIT I SINGLE TARGET
178     LOOP  "ON PRIORITY L
179     FOR I = 1 TO 10 DO   "REINITIALIZE CONTROL VARS FOR NEXT REP
180       FOR W = 1 TO 2 DO
181         LET A(I,W) = 0
182         LET B(I,W) = 0
183         LET ENGAGE(I) = 0
184       LOOP
185     LOOP
186     LET NW1T = NW1   " REINITIALIZE ROUNDS FOR NEXT REP
187     LET NW2T = NW2
188     LET NW3T = NW3
189   LOOP "ON K = 1 TO N
190   PRINT 1 LINE WITH N THUS
     FOR ***** REPLICATIONS, AVERAGES ARE:
192   FOR I = 1 TO 10 DO
193     PRINT 1 LINE WITH I,CA(I,1)/N,I,PSINGLE(I,1),I,CA(I,2)/N,I,PSINGLE(I,2) THUS
    A(**,1) = *.***  PA(**,2) = *.***   A(**,2) = *.***  PA(**,2) = *.***
```

```
195    PRINT 1 LINE WITH I,CB(I,1)/N,I,PAGGR(I,1),I,CB(I,2)/N,I,PAGGR(I,2) THUS
   B(**,1) = *.***   PB(**,1) = *.***    B(**,2) = *.***   PB(**,2) = *.***
197  LOOP
198  PRINT 1 LINE THUS
   SINGLE TARGETS:
200  FOR I = 1 TO 10 DO
201    FOR W = 1 TO 2 DO
202      IF CROUND(I,W) = 0
203        IF PROUND(I,W) NE 0
204          PRINT 1 LINE WITH I,W,PROUND(I,W), I,W,CROUND(I,W) THUS
----ERROR----   PROUND(**,*)= ***   CROUND(**,*)= ***
206        ALWAYS
207        LET CROUND(I,W) = 1
208      ALWAYS
209    LOOP
210    PRINT 1 LINE WITH I,PROUND(I,1)/CROUND(I,1),I,PROUND(I,2)/CROUND(I,2)
   THUS
   PROUND(**,1) = *.***   PROUND(**,2) = *.***
212  LOOP
213  PRINT 2 LINES THUS

   AGGREGATE TARGETS:
216  FOR I = 1 TO 10 DO
217    FOR W = 1 TO 2 DO
218      IF CROUND(I+10,W) = 0
219        IF PROUND(I+10,W) NE 0
220          PRINT 1 LINE WITH I+10,W,PROUND(I+10,W), I+10,W,CROUND(I+10,W) THUS
----ERROR----   PROUND(**,*)= ***   CROUND(**,*)= ***
222        ALWAYS
223        LET CROUND(I+10,W) = 1
224      ALWAYS
225    LOOP
226    PRINT 1 LINE WITH I,IPAIR(I,1),PROUND(I+10,1)/CROUND(I+10,1),
227        I,IPAIR(I,2),PROUND(I+10,2)/CROUND(I+10,2) THUS
   PROUND(**,**) = *.***   PROUND(**,**) = *.***
229  LOOP
230 END
```

Input Data

WEAPON # 1 :

```
PSINGLE( 1,1) =  .129 FOR J =  2,    PAGGR( 1,1) =  .048
PSINGLE( 2,1) =  .263 FOR J =  3,    PAGGR( 2,1) =  .108
PSINGLE( 3,1) =  .353 FOR J =  2,    PAGGR( 3,1) =  .108
PSINGLE( 4,1) =  .218 FOR J =  3,    PAGGR( 4,1) =  .114
PSINGLE( 5,1) =  .326 FOR J =  6,    PAGGR( 5,1) =  .286
PSINGLE( 6,1) =  .448 FOR J =  7,    PAGGR( 6,1) =  .119
PSINGLE( 7,1) =  .388 FOR J =  6,    PAGGR( 7,1) =  .119
PSINGLE( 8,1) =  .128 FOR J =  7,    PAGGR( 8,1) =  .044
PSINGLE( 9,1) =  .234 FOR J =  3,    PAGGR( 9,1) =  .038
PSINGLE(10,1) =  .200 FOR J = 10,    PAGGR(10,1) = 0.
```

WEAPON # 2 :

PSINGLE(1,2) = .080 FOR J = 2,	PAGGR(1,2) = .045	
PSINGLE(2,2) = .164 FOR J = 3,	PAGGR(2,2) = .132	
PSINGLE(3,2) = .323 FOR J = 2,	PAGGR(3,2) = .132	
PSINGLE(4,2) = .135 FOR J = 3,	PAGGR(4,2) = .142	
PSINGLE(5,2) = .195 FOR J = 6,	PAGGR(5,2) = .242	
PSINGLE(6,2) = .313 FOR J = 7,	PAGGR(6,2) = .210	
PSINGLE(7,2) = .274 FOR J = 6,	PAGGR(7,2) = .210	
PSINGLE(8,2) = .052 FOR J = 7,	PAGGR(8,2) = .072	
PSINGLE(9,2) = .169 FOR J = 6,	PAGGR(9,2) = .045	
PSINGLE(10,2) = .200 FOR J = 10,	PAGGR(10,2) = 0.	

FPREF(1,1) = 1	FPREF(1,2) = 1	FPREF(1,3) = 0
FPREF(2,1) = 1	FPREF(2,2) = 1	FPREF(2,3) = 0
FPREF(3,1) = 0	FPREF(3,2) = 1	FPREF(3,3) = 0
FPREF(4,1) = 1	FPREF(4,2) = 0	FPREF(4,3) = 1
FPREF(5,1) = 1	FPREF(5,2) = 1	FPREF(5,3) = 1
FPREF(6,1) = 0	FPREF(6,2) = 1	FPREF(6,3) = 1
FPREF(7,1) = 0	FPREF(7,2) = 0	FPREF(7,3) = 1
FPREF(8,1) = 1	FPREF(8,2) = 1	FPREF(8,3) = 1
FPREF(9,1) = 0	FPREF(9,2) = 1	FPREF(9,3) = 1
FPREF(10,1) = 0	FPREF(10,2) = 0	FPREF(10,3) = 1

FPREF(11, 2,1) = 1	FPREF(11, 2,2) = 1	FPREF(11, 2,3) = 0
FPREF(12, 3,1) = 0	FPREF(12, 3,2) = 1	FPREF(12, 3,3) = 0
FPREF(13, 2,1) = 0	FPREF(13, 2,2) = 1	FPREF(13, 2,3) = 0
FPREF(14, 3,1) = 0	FPREF(14, 3,2) = 0	FPREF(14, 3,3) = 0
FPREF(15, 6,1) = 0	FPREF(15, 6,2) = 1	FPREF(15, 6,3) = 1
FPREF(16, 7,1) = 0	FPREF(16, 7,2) = 0	FPREF(16, 7,3) = 1
FPREF(17, 6,1) = 0	FPREF(17, 6,2) = 0	FPREF(17, 6,3) = 1
FPREF(18, 7,1) = 0	FPREF(18, 7,2) = 0	FPREF(18, 7,3) = 1
FPREF(19, 3,1) = 0	FPREF(19, 3,2) = 1	FPREF(19, 6,3) = 1
FPREF(20,10,1) = 0	FPREF(20,10,2) = 0	FPREF(20,10,3) = 0

PRI(1) = 3	PRI(11) = 3
PRI(2) = 3	PRI(12) = 3
PRI(3) = 3	PRI(13) = 3
PRI(4) = 2	PRI(14) = 3
PRI(5) = 2	PRI(15) = 2
PRI(6) = 2	PRI(16) = 2
PRI(7) = 2	PRI(17) = 2
PRI(8) = 2	PRI(18) = 2
PRI(9) = 2	PRI(19) = 2
PRI(10) = 1	PRI(20) = 1

Output

FOR 5000 REPLICATIONS, AVERAGES ARE: "SHOWN TO ESTABLISH HOW CLOSE THE REPLICATIONS ARE TO THE THEORETICAL RESULT

A(1,1) = .130	PA(1,2) = .129	A(1,2) = .076	PA(1,2) = .080
B(1,1) = .052	PB(1,1) = .048	B(1,2) = .045	PB(1,2) = .045
A(2,1) = .255	PA(2,2) = .263	A(2,2) = .170	PA(2,2) = .164
B(2,1) = .107	PB(2,1) = .108	B(2,2) = .135	PB(2,2) = .132
A(3,1) = .359	PA(3,2) = .353	A(3,2) = .305	PA(3,2) = .323
B(3,1) = .113	PB(3,1) = .108	B(3,2) = .140	PB(3,2) = .132
A(4,1) = .217	PA(4,2) = .218	A(4,2) = .139	PA(4,2) = .135
B(4,1) = .115	PB(4,1) = .114	B(4,2) = .138	PB(4,2) = .142
A(5,1) = .325	PA(5,2) = .326	A(5,2) = .191	PA(5,2) = .195
B(5,1) = .290	PB(5,1) = .286	B(5,2) = .247	PB(5,2) = .242
A(6,1) = .448	PA(6,2) = .448	A(6,2) = .317	PA(6,2) = .313
B(6,1) = .121	PB(6,1) = .119	B(6,2) = .206	PB(6,2) = .210
A(7,1) = .390	PA(7,2) = .388	A(7,2) = .266	PA(7,2) = .274
B(7,1) = .116	PB(7,1) = .119	B(7,2) = .210	PB(7,2) = .210
A(8,1) = .126	PA(8,2) = .128	A(8,2) = .052	PA(8,2) = .052
B(8,1) = .042	PB(8,1) = .044	B(8,2) = .076	PB(8,2) = .072
A(9,1) = .242	PA(9,2) = .234	A(9,2) = .162	PA(9,2) = .169
B(9,1) = .031	PB(9,1) = .038	B(9,2) = .043	PB(9,2) = .045
A(10,1) = .201	PA(10,2) = .200	A(10,2) = .201	PA(10,2) = .200
B(10,1) = 0.	PB(10,1) = 0.	B(10,2) = 0.	PB(10,2) = 0.

(1) Run #1

WEAPONS AVAILABLE:
1 KT, FIRE UNIT 1 : 1   1 KT, FIRE UNIT 2 : 1   10 KT, FIRE UNIT 3 : 1

SINGLE TARGETS:

PROUND(1,1) = .498	PROUND(1,2) = 0.
PROUND(2,1) = .434	PROUND(2,2) = 0.
PROUND(3,1) = .368	PROUND(3,2) = 0.
PROUND(4,1) = 1.000	PROUND(4,2) = .384
PROUND(5,1) = .807	PROUND(5,2) = .377
PROUND(6,1) = .974	PROUND(6,2) = .454
PROUND(7,1) = 0.	PROUND(7,2) = .420
PROUND(8,1) = .565	PROUND(8,2) = .199
PROUND(9,1) = .528	PROUND(9,2) = .176
PROUND(10,1) = 0.	PROUND(10.2) = 1.000

AGGREGATE TARGETS:

PROUND(1, 2) = .544	PROUND(1, 2) = 0.
PROUND(2, 3) = .368	PROUND(2, 3) = 0.
PROUND(3, 2) = .369	PROUND(3, 2) = 0.
PROUND(4, 3) = 0.	PROUND(4, 3) = 0.
PROUND(5, 6) = 1.000	PROUND(5, 6) = .788
PROUND(6, 7) = 0.	PROUND(6, 7) = .790
PROUND(7, 6) = 0.	PROUND(7, 6) = .629
PROUND(8, 7) = 0.	PROUND(8, 7) = .454
PROUND(9, 3) = .684	PROUND(9, 6) = .430
PROUND(10,10) = 0.	PROUND(10,10) = 0.

(2) *Run #2*

WEAPONS AVAILABLE:
1 KT, FIRE UNIT 1 : 2   1 KT, FIRE UNIT 2 : 2   10 KT, FIRE UNIT 3 : 2

SINGLE TARGETS:

PROUND(1,1) = .920   PROUND(1,2) = 0.
PROUND(2,1) = .853   PROUND(2,2) = 0.
PROUND(3,1) = .878   PPOUND(3,2) = 0.
PROUND(4,1) = 1.000   PROUND(4,2) = .835
PROUND(5,1) = 1.000   PROUND(5,2) = .850
PROUND(6,1) = 1.000   PROUND(6,2) = .895
PROUND(7,1) = 0.   PROUND(7,2) = .854
PROUND(8,1) = .954   PROUND(8,2) = .665
PROUND(9,1) = 1.000   PROUND(9,2) = .632
PROUND(10,1) = 0.   PROUND(10,2) = 1.000

AGGREGATE TARGETS:
PROUND(1, 2) = .873   PROUND(1, 2) = 0.
PROUND(2, 3) = .860   PROUND(2, 3) = 0.
PROUND(3, 2) = .876   PROUND(3, 2) = 0.
PROUND(4, 3) = 0.   PROUND(4, 3) = 0.
PROUND(5, 6) = 1.000   PROUND(5, 6) = 1.000
PROUND(6, 7) = 0.   PROUND(6, 7) = 1.000
PROUND(7, 6) = 0.   PROUND(7, 6) = .950
PROUND(8, 7) = 0.   PROUND(8, 7) = .887
PROUND(9, 3) = 1.000   PROUND(9, 6) = .821
PROUND(10,10) = 0.   PROUND(10,10) = 0.

(3) *Run #3*

WEAPONS AVAILABLE:
1 KT, FIRE UNIT 1 : 4   1 KT, FIRE UNIT 2 : 2   10 KT, FIRE UNIT 3 : 4

SINGLE TARGETS:

PROUND(1,1) = 1.000   PROUND(1,2) = 0.
PROUND(2,1) = .998   PROUND(2,2) = 0.
PROUND(3,1) = .878   PROUND(3,2) = 0.
PROUND(4,1) = 1.000   PROUND(4,2) = 1.000
PROUND(5,1) = 1.000   PROUND(5,2) = 1.000
PROUND(6,1) = 1.000   PROUND(6,2) = .997
PROUND(7,1) = 0.   PROUND(7,2) = .999
PROUND(8,1) = 1.000   PROUND(8,2) = .986
PROUND(9,1) = 1.000   PROUND(9,2) = .986
PROUND(10,1) = 0.   PROUND(10,2) = 1.000

AGGREGATE TARGETS:

PROUND(1, 2) = 1.000	PROUND(1, 2) = 0.
PROUND(2, 3) = .860	PROUND(2, 3) = 0.
PROUND(3, 2) = .876	PROUND(3, 2) = 0.
PROUND(4, 3) = 0.	PROUND(4, 3) = 0.
PROUND(5, 6) = 1.000	PROUND(5, 6) = 1.000
PROUND(6, 7) = 0.	PROUND(6, 7) = 1.000
PROUND(7, 6) = 0.	PROUND(7, 6) = 1.000
PROUND(8, 7) = 0.	PROUND(8, 7) = 1.000
PROUND(9, 3) = 1.000	PROUND(9, 6) = 1.000
PROUND(10,10) = 0.	PROUND(10,10) = 0.

Section IV - CALCULATING PROBABILITIES OF ENGAGEMENT AND DEFEAT

Probabilities of Engagement

Recall the following definitions:

$$p_{engage}(i, s, w) = p_{avail}(i) \cdot p_{shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) \cdot p_{round}(w \mid a_i)$$

$$= p_{engage}(i, s \mid w) \cdot p_{round}(w \mid a_i)$$

$$p_{engage}(i, \overline{s}, w) = p_{avail}(i) \cdot p_{no\ shift}(i \mid w, a_i, a\overline{g}g_i) \cdot \overline{p}_{aggr}(i \mid w, a_i) \cdot p_{round}(w \mid a_i)$$

$$= p_{engage}(i, \overline{s} \mid w) \cdot p_{round}(w \mid a_i)$$

$$p_{engage}(i, w) = p_{engage}(i \mid w) \cdot p_{round}(w \mid a_i)$$

$$p_{engage}(ij, s, w) = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij}) \cdot p_{round}(w \mid a_{ij})$$

$$= p_{engage}(ij, s \mid w) \cdot p_{round}(w \mid a_{ij})$$

$$p_{engage}(ij, \overline{s}, w) = p_{avail}(i) \cdot p_{avail}(j) \cdot p_{no\ shift}(ij \mid w, a_{ij}, agg_{ij}) \cdot p_{aggr}(ij \mid w, a_{ij})$$

$$\cdot p_{round}(w \mid a_{ij})$$

$$= p_{engage}(ij, \overline{s} \mid w) \cdot p_{round}(w \mid a_{ij})$$

$$p_{engage}(ij, w) = p_{engage}(ij \mid w) \cdot p_{round}(w \mid a_{ij})$$

In this case, the values PROUND(I, W) from the Monte Carlo estimation equal $p_{round}(w \mid a_i)$ and the values PROUND(I, J, W) from the Monte Carlo estimation equal $p_{round}(w \mid a_{ij})$

Run #1

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(i, \overline{s}, w_1)$	.043	.076	.086	.109	.175	.272	0	.054	.069	0
$p_{engage}(i, s, w_1)$	.021	.038	.043	.109	.088	.164	0	.018	.055	0
$p_{engage}(i, \overline{s}, w_2)$	0	0	0	.023	.066	.032	.029	.006	.005	.180
$p_{engage}(i, s, w_2)$	0	0	0	.020	.041	.022	.021	.002	.002	.020

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(ij, \overline{s}, w_1)$	.013	.026	.027	0	.176	0	0	0	.016	0
$p_{engage}(ij, s, w_1)$	.013	.013	.013	0	.110	0	0	0	.008	0
$p_{engage}(ij, \overline{s}, w_2)$	0	0	0	0	.087	.077	.062	.015	.006	0
$p_{engage}(ij, s, w_2)$	0	0	0	0	.110	.098	.098	.032	.015	0

Run #2

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(i, \overline{s}, w_1)$	.079	.149	.206	.109	.217	.280	0	.092	.130	0
$p_{engage}(i, s, w_1)$	.040	.075	.104	.109	.109	.168	0	.031	.104	0
$p_{engage}(i, \overline{s}, w_1)$	0	0	0	.051	.150	.063	.060	.021	.017	.180
$p_{engage}(i, s, w_2)$	0	0	0	.044	.094	.044	.042	.008	.007	.020

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(ij, \overline{s}, w_1)$	.021	.062	.063	0	.176	0	0	0	.023	0
$p_{engage}(ij, s, w_1)$	.023	.031	.032	0	.110	0	0	0	.011	0
$p_{engage}(ij, \overline{s}, w_1)$	0	0	0	0	.110	.098	.093	.028	.012	0
$p_{engage}(ij, s, w_2)$	0	0	0	0	.132	.112	.106	.035	.025	0

Run #3

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(i, \overline{s}, w_1)$	.086	.175	.206	.109	.217	.280	0	.096	.130	0
$p_{engage}(i, s, w_1)$	.043	.088	.104	.109	.109	.168	0	.032	.104	0
$p_{engage}(i, \overline{s}, w_1)$	0	0	0	.061	.176	.070	.070	.032	.027	.180
$p_{engage}(i, s, w_2)$	0	0	0	.053	.110	.049	.049	.012	.011	.020

Unit	1	2	3	4	5	6	7	8	9	10
$p_{engage}(ij, \overline{s}, w_1)$	.024	.062	.-63	0	.176	0	0	0	.023	0
$p_{engage}(ij, s, w_1)$	.024	.031	.032	0	.110	0	0	0	.011	0
$p_{engage}(ij, \overline{s}, w_1)$	0	0	0	0	.110	.098	.098	.032	.015	0
$p_{engage}(ij, s, w_2)$	0	0	0	0	.132	.112	.112	.040	.030	0

Probabilities of Defeat

Recall the following definitions:

$P_{defeat}(i, w)$

$$:= P_{defeat}(i \mid w, a_i, s, a\overline{g}g_i) \cdot P_{engage}(i, s, w) + P_{defeat}(i \mid w, a_i, \overline{s}, a\overline{g}g_i) \cdot P_{engage}(i, \overline{s}, w)$$

$$+ P_{defeat}(i \mid w, a_{ij}, s, agg_{ij}) \cdot P_{engage}(ij, s, w) + P_{defeat}(i \mid w, a_{ij}, \overline{s}, agg_{ij}) \cdot P_{engage}(ij, \overline{s}, w)$$

$$P_{defeat}(i) = \sum_w P_{defeat}(i, w)$$

Suppose we have the following conditional probabilities of defeat:

Unit	1	2	3	4	5	6	7	8	9	10
$P_{defeat}(i \mid w_1, \overline{s}, \ldots)$	.75	.75	.75	.65	.65	.65	.65	.70	.70	-
$P_{defeat}(i \mid w_1, s, \ldots)$	.40	.50	.60	.35	.40	.45	.60	.35	.50	-
$P_{defeat}(i \mid w_2, \overline{s}, \ldots)$	.90	.90	.90	.80	.80	.80	.80	.85	.85	1.0
$P_{defeat}(i \mid w_2, s, \ldots)$	.55	.40	.50	.40	.45	.55	.70	.30	.55	.60

Unit	1	2	3	4	5	6	7	8	9	10
$P_{defeat}(ij \mid w_1, \overline{s}, \ldots)$	.60	.55	.55	.50	.55	.45	.45	.55	.50	-
$P_{defeat}(ij \mid w_1, s, \ldots)$	.25	.30	.30	.20	.25	.35	.35	.20	.25	-
$P_{defeat}(ij \mid w_2, \overline{s}, \ldots)$	.70	.65	.65	.60	.70	.65	.65	.70	.70	-
$P_{defeat}(ij \mid w_2, s, \ldots)$	.15	.25	.25	.25	.35	.25	.25	.40	.35	-

Thus for run #1,

$$P_{defeat}(1, w_1) = (.043)(.75) + (.021)(.40) + (.013)(.60) + (.013)(.25) = .052, \text{ etc.}$$

Unit	1	2	3	4	5	6	7	8	9	10
$P_{defeat}(i, w_1)$ Run 1	.056	.096	.110	.109	.295	.251	0	.044	.088	0
$P_{defeat}(i, w_1)$ Run 2	.099	.196	.268	.109	.331	.258	0	.075	.161	0
$P_{defeat}(i, w_1)$ Run 3	.109	.222	.268	.109	.331	.258	0	.078	.161	0
$P_{defeat}(i, w_2)$ Run 1	0	0	0	.026	.169	.110	.096	.023	.014	.192
$P_{defeat}(i, w_2)$ Run 2	0	0	0	.058	.290	.166	.164	.054	.035	.192
$P_{defeat}(i, w_2)$ Run 3	0	0	0	.070	.319	.175	.182	.069	.050	.192

This yields the following values for $p_{defeat}(i)$:

Unit	1	2	3	4	5	6	7	8	9	10
$p_{defeat}(i)$ Run 1	.056	.096	.110	.135	.464	.361	.096	.067	.102	.192
$p_{defeat}(i)$ Run 2	.099	.196	.268	.167	.621	.424	.164	.129	.196	.192
$p_{defeat}(i)$ Run 3	.109	.222	.268	.179	.650	.433	.182	.147	.211	.192

APPENDIX I

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